Berkeley Math Circle Monthly Contest # 3
Due November , 2001

1. Find all $f : \mathbf{Z} \rightarrow \mathbf{Z}$ such that
$$f(x) + f(y) = f(\frac{x+y}{2}) + f(3x).$$

2. For $a, b, c \geq 0$, find the minimum of
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

3. A group of goblins decide to participate in a game of ritual slaughter. The goblins have a well defined ranking based on who is the most vile. Each round of the game the most vile living goblin recommends killing some subset of the current living goblins. The goblins then vote on the plan. If the plan wins a majority vote (greater than 50% of the goblins vote for it) those goblins selected in the plan are slaughtered and the game ends. If the plan fails the vote then the goblin who suggested the plan is killed and the next round proceeds where in the new most vile living goblin suggests another plan. Each goblin know his or her vileness ranking and the ranking of all other goblins in the group and all goblins are perfectly logical and know all the other goblins are perfectly logical. The goblins vote according to the following list of rules. If two rules conflict the higher rule listed first takes priority.

    1. The goblin wants to survive the game

    2. The goblin wants to kill as many other goblins as possible

    3. All other things being equal the goblin prefers to move vile ranking goblins. No one likes those in power. Note, if two sets of goblins are compared for vileness the most vile goblins are compared first, then if equal the next most vile, etc...

    For each number of starting goblins N, determine how many and which goblins are killed.

4. Consider two points, A and B. Let $|AB| = 100$. Construct the line AB with a straightedge of length 10. You cannot use a compass. The straightedge has no markings.

5. Consider $\triangle ABC$ with $AC < AB$. Let H, M, and D be the feet of the altitude, median, and angle bisector from A to BC, respectively. Let X be the foot of the perpendicular from D onto AB. Let Z be on segment AC such that $CZ = \frac{DB*CB}{2AB}$. Let ω_1 be the circle through $A, C,$ and H and ω_2 be the circle through A, B and M. Let Y be the intersection of ω_1 and ω_2. Prove that $X, Y, Z,$ and A lie on a circle.