Berkeley Monthly Contest #2 Solutions

1. We will prove that Oaz wins regardless of strategy by contradiction. Suppose that there is a game when Andrew wins. Then there are two numbers, $a$ and $b$, such that $a + b = 101$, and $\gcd(a, b) = d > 1$. Then $d|a$ and $d|b$, so $a = dx$ and $b = dy$ for positive integers $x$ and $y$. $a + b = dx + dy = d(x + y)$, so $d|101$. Since 101 is prime, $d = 1$ or 101. However, by the assumption $d \neq 1$ so $d = 101$. Then $x + y = 1$, which is impossible. Thus our original assumption was false, and Oaz must always win.

2. We will first show that on a circle with a continuous funtion defined on the circumference there are two diametrically opposite points with the same temperature. Let $T(x)$ be the value of the function at point $x$, and $A(x)$ to be the point diametrically opposite $x$. Then $T(A(x)) > T(x)$ or $T(A(x)) < T(x)$ or $T(A(x)) = T(x)$. If the third is true we are done. Suppose $T(A(x)) \neq x$. Then, let $T(x) - T(A(x)) > 0$. Then $T(A(A(x))) - T(A(x)) > 0$. Since $T$ is continuous (by definition) and $A$ is continuous the difference above is also continuous. Since as $x$ moves to $A(x)$ the sign of the difference changes there must be a point when it is 0. Thus there is a $y$ between $x$ and $A(x)$ such that $T(y) - T(A(y)) = 0$, as required.
   Since the temperature on the planet is a continuous function, it will be continuous on its restriction onto a great circle. On this great circle there will be two diametrically opposite points with the same temperature.

3. Case 1: $m = 0$.
   If $m = 0$ then the equation is $x^2(x^2+x+1)$ so the roots of the expression are $x_1 = x_2 = 0$, $x_3 = \frac{-1+\sqrt{-3}}{2}$, and $x_4 = \frac{-1-\sqrt{-3}}{2}$.
   Case 2: $m \neq 0$.
   Since $m \neq 0$ 0 is not a root of the equation. Because of this we can divide the entire expression by $x^2$, which gives us
   $$x^2 + x + 1 + \frac{m}{x} + \frac{m^2}{x^2} = 0.$$
   By using the substitution $y = x + \frac{m}{x}$ we get
   $$y^2 + y + 1 - 2m = 0$$

. Using the quadratic formula on the above expression yields $y_1 = \frac{-1+\sqrt{8m-3}}{2}$ and $y_2 = \frac{-1-\sqrt{8m-3}}{2}$. Substituting that for $y$ yields

$$x^2 - \frac{-1+\sqrt{8m-3}}{2}x + m = 0 \text{ and } x^2 - \frac{-1-\sqrt{8m-3}}{2}x + m = 0.$$

With the quadratic formula we get the four roots of the original expression:

$$x_1 = \frac{-1+\sqrt{8m-3}+i\sqrt{2+8m+2\sqrt{8m3}}}{4}$$

$$x_2 = \frac{-1+\sqrt{8m-3}-i\sqrt{2+8m+2\sqrt{8m3}}}{4}$$

$$x_3 = \frac{-1-\sqrt{8m-3}+i\sqrt{2+8m-2\sqrt{8m-3}}}{4}$$

$$x_4 = \frac{-1-\sqrt{8m-3}-i\sqrt{2+8m-2\sqrt{8m-3}}}{4}$$

In this problem, the omission of the first case does not affect the answers, since plugging in $m = 0$ gives the correct answers. However, it is still incorrect to omit the first case, since in the case $m = 0$ we cannot divide by $x^2$.

4. Since $ABCD$ is cyclic we know that $\angle A = \pi - \angle C$ and $\angle B = \pi - \angle D$. Thus $\sin(A) = \sin(C)$ and $\sin(B) = \sin(D)$. By law of sines, $\frac{AC}{\sin(B)} = 2R$, where $R$ is the radius of the given circle (since it is circumscribed around the quadrilateral). Also, $\frac{BD}{\sin A} = 2R$. So

$$\frac{AC}{\sin B} = \frac{BD}{\sin A} \Rightarrow \frac{AC}{BD} = \frac{\sin B}{\sin A}$$

$$= \frac{\sin B * \text{area}}{\sin A * \text{area}}$$

$$= \frac{\sin B}{\sin A} * \frac{.5\sin A * AB * AD + .5\sin A * CB * CD}{.5\sin B * BC * BA + .5\sin B * DC * DA}$$

$$= \frac{\sin B}{\sin A} * \frac{\sin A}{\sin B} * \frac{CD * CB + AB * AD}{BC * BA + DC * DA}$$

$$= \frac{CD * CB + AB * AD}{BC * BA + DC * DA}.$$

Another solution follows a similar line of reasoning with the law of sines replaced by the formula

$$\text{area}(\triangle ABC) = \frac{AB * BC * CA}{4R}.$$

5. Poor Neil's soul is destined for eternal slavery to the nefarious Inna! By Bertrand's Postulate, for any integer $k > 1$, there exists a prime $p$ such that $k < p < 2k$. Let $k = n^2/2$ ($n$ even) or $(n^2+1)/2$ ($n$ odd). Now $p \leq 2(n^2)/2 = n^2$ and clearly $p \neq n^2$ so $p < n^2$, while $2p > 2k >= 2(n^2)/2 = n^2$) Thus, exactly one of $\{1, 2, ..., n^2\}$ is a multiple of $p$. Therefore the row and column containing $p$ are the unique row and column with $p$ dividing the product of the elements. Clearly then, with $n > 1$ there are two rows with differing products regardless of the configuration. Neil is forever Inna's prisoner!