Berkeley Math Circle 2000-2001

Monthly Contest #8 — Solutions

1. Every point of the plane is colored either red or blue. Prove that there exists an equilateral triangle all of whose vertices are the same color.

   Solution: Suppose that no such triangle exists; we will obtain a contradiction. Let ABC be any equilateral triangle of side 1; then, by assumption, two vertices of $\triangle ABC$ are one color and the other vertex is the second color. Without loss of generality, we may suppose A and B are red, and C is blue. Construct equilateral $\triangle ABD$ ($D \neq C$); then D must also be blue. Extend ray AB to point E such that $BE = 1$, and note that $\triangle CDE$ is equilateral, with $CD = CE = DE = \sqrt{3}$. Since C and D are both blue, E must be red.

   But if we draw equilateral $\triangle BEF$, with F and C on the same side of AB, then F must be blue (since B, E are red). Now complete the equilateral triangle CFG ($G \neq B$). We see that C and F are blue, so G is red. However, $\triangle AEG$ is also equilateral with A and E red, so G should be blue. This is our contradiction, so our assumption — that no monochrome equilateral triangles existed — must be false.

2. The UC Berkeley math department is about to move into a new, one-story building consisting of a 2001×2001 square grid of rooms. They would like to install doors between adjacent rooms so that each room has exactly two doors. Prove that this cannot be done.

   Solution: Suppose that it can be done. Color the building in checkerboard fashion, and suppose that we obtain a white rooms and b black rooms. Each door connects a white room with a black room. So, if we consider, for each white room, the number of doors adjoining it, we will count each door exactly once. Since every room is to have 2 doors, the total number of doors will be $2a$ by this count. But similarly, if we consider, for each black room, the number of adjoining doors, each door will be counted once, so that the total number of doors is also equal to $2b$. So, $2a = 2b$, or $a = b$. It follows that the total number of rooms is $a + b = 2a$, an even number. But we also know the number of rooms is 2001^2, an odd number — contradiction. Hence, the desired condition cannot be met.

3. The manager of Chez Gastropod wants to write a menu consisting of 15 dishes. A "meal" is defined to be a subset of this menu (possibly empty), but some meals are legal and others are not. The manager may choose which meals are legal, but there is a requirement that the intersection of any two legal meals should still be legal. He wants there to be exactly 2001 legal meals. Can he do it?

   Solution: The answer is yes. Take any arbitrary 15-element set (menu), and call a collection of subsets (meals) "valid" if the intersection of any two sets in the collection is again in the collection. Thus, the objective is to show that there exists a valid collection containing exactly 2001 sets. We will show, by downward induction, that there exists a valid collection with exactly n sets for each $n, 0 \leq n \leq 2^{15} = 32768$.

   The base case $n = 2^{15}$ is clear: the collection of all subsets of the menu is certainly valid. For the induction step, suppose that there is a valid collection C of n subsets ($1 \leq n \leq 32768$); we will show that there is a valid collection of $n - 1$ subsets. Choose a subset in C of maximum possible size, and remove it; let C' denote the remaining collection, so that it consists of $n - 1$ subsets. We claim C' is still valid. Indeed, if $S, T \in C'$, then the removed subset cannot be contained within either S or T (because of maximality), hence it certainly cannot equal their intersection. But $S \cap T$ was in C; hence, it is also in C', as needed. Thus, the claim holds. Now, let $n = 2001$, and the problem is solved.

4. Given a line segment AB, construct a segment half as long as AB using only a compass. Construct a segment one-third as long as AB using only a compass.

   Solution: First, we provide (part of) an algorithm for circular inversion. Suppose we are given point O and a circle centered at O of some radius r. If P is a point outside the circle, we wish to construct point Q on ray OP, satisfying $OP \cdot OQ = r^2$. Let the circle centered at P, with radius OP, intersect the given circle centered at O at points X and Y. Then let the circle with center X, radius r (which is constructible since $r = OX$), and the circle with center Y, radius r, intersect at O and Q. We claim this point Q is what we want. Indeed, it is clear from symmetry that OP is the perpendicular bisector of XY. Since $XQ = r = YQ$, Q lies on this bisector as well — that is, on line OP. Moreover, let H denote the intersection of XY with OP; then $PH < PX = PO \Rightarrow H$ lies on ray OP, and, since Q is the reflection of O across XY, Q will lie on the opposite side of H from O. This shows that Q lies on ray OP, not just on line OP. Finally, observe that $PX = PO$ and $XO = XQ \Rightarrow \angle OXP = \angle POX = \angle QOX = \angle XQO$, so, by equal angles, $\triangle POX \sim \triangle XQO$. Thus, $OP/OX = QX/QO \Rightarrow OP \cdot OQ = OX \cdot QX = r^2$, as needed.

Now that this is done, we return to the original problem. By scaling, assume $AB = 1$. By drawing circles centered at A and B of radius 1, and letting C be one of their intersection points, we obtain an equilateral $\triangle ABC$. Similarly, we successively construct equilateral triangles BCD, BDE (with $D \neq A, E \neq C$). We have $\angle ABE = \angle ABC + \angle CBD + \angle DBE = 3(\pi/3) = \pi$, so A, B, E are collinear, and $AE = AB + BE = 2$. Then, since we have drawn the circle with center A and radius 1, we can invert E across it according to the paragraph above, obtaining F such that $AF = 1/2$. In fact, F lies on the given segment AB, so the segment AF is fully drawn.

Similarly, to construct a segment of length $1/3$, let AB be given; extend AB to E as above so that $BE = 1$, and then repeat the process, extending BE to G so that $EG = 1$. Then $AG = 3$, and inverting G across our circle (center A, radius 1) will give what we need.

5. Let $a_1 = 3$ and define $a_{n+1} = (3a_n^2 + 1)/2 - a_n$ for $n \geq 1$. If n is a power of 3, prove that a_n is divisible by n.

   Solution: The main trick is finding a closed-form expression for a_n, which requires some experimentation. We will show that $a_n = (2^{2^n+1} + 1)/3$ for all n by induction. It is easy to check that the formula holds for $n = 1$. And if it holds for some n, then

$$\begin{aligned}
a_{n+1} &= \frac{3a_n^2+1}{2} - a_n \\
&= \frac{1}{2}\Big(3\Big[\frac{2^{2^n+1}+1}{3}\Big]^2 + 1\Big) - \frac{2^{2^n+1}+1}{3} \\
&= \frac{1}{2}\Big(\frac{(2^{2^n+1})^2 + 2(2^{2^n+1}) + 1}{3} + 1\Big) - \frac{2^{2^n+1}+1}{3} \\
&= \frac{1}{2}\Big(\frac{2^{2^{n+1}+2} + 2(2^{2^n+1}) + 4}{3}\Big) - \frac{2^{2^n+1}+1}{3} \\
&= \frac{2^{2^{n+1}+1} + 2^{2^n+1} + 2}{3} - \frac{2^{2^n+1}+1}{3} \\
&= \frac{2^{2^{n+1}+1}+1}{3},
\end{aligned}$$

giving the induction step.

Now, by Euler's theorem, 3^k divides $2^{2\cdot 3^{k-1}} - 1$, for any nonnegative integer k, since $\phi(3^k)$ (i.e. the number of integers in $\{1, 2, \ldots, 3^k\}$ relatively prime to 3^k) equals $2 \cdot 3^{k-1}$. But notice that $2^{2\cdot 3^{k-1}} - 1 = (2^{3^{k-1}} - 1)(2^{3^{k-1}} + 1)$, and 3^{k-1} is odd $\Rightarrow 2^{3^{k-1}} \equiv 2 \pmod 3 \Rightarrow 2^{3^{k-1}} - 1$ is relatively prime to 3^k, so, in fact, 3^k divides $2^{3^{k-1}} + 1$. Also, for any integers c, d with d odd, $2^c + 1 \mid 2^{cd} + 1$. We conclude that $3^k \mid 2^a + 1$ whenever $3^{k-1} \mid a$ and a is odd.

Applying this result twice in succession, we find that $3^k \mid 2^{3^k} + 1$ and then that $3^{k+1} \mid 2^{2^{3^k}+1} + 1$, so that $3^k \mid (2^{2^{3^k}+1} + 1)/3 = a_{3^k}$ for any integer $k \geq 0$, and this is what we wanted to prove.