Berkeley Math Circle 2000-2001

Monthly Contest #7 — Solutions

1. Show that there exist infinitely many natural numbers n with the following property: the sum of all the positive divisors of n, *excluding* n itself, equals $n + 12$.

   Solution: Let p be any prime number greater than 3; we show that $n = 6p$ has the desired property. The positive divisors of $6p = 2 \cdot 3 \cdot p$ are $1, 2, 3, p, 2 \cdot 3, 2 \cdot p, 3 \cdot p$, and $2 \cdot 3 \cdot p$. The sum of all the factors other than $6p$ is equal to $6p + 12$, as needed.

   It is well known that there are infinitely many prime numbers. Since each value of p gives a different value for $n = 6p$, we obtain infinitely many values for n.

2. 5 married couples gather at a party. As they come in and greet each other, various people exchange handshakes — but, of course, people never shake hands with themselves or with their own respective spouses. At the end of the party, one woman goes around asking people how many hands they shook, and she gets nine different answers. How many hands did she herself shake?

   Solution: Suppose that there were n couples, and the woman asked all $2n - 1$ other attendees how many hands they shook and received $2n - 1$ different answers. We will show that she herself shook $n - 1$ hands; hence, in our particular case, the answer is 4.

   We work by induction. When $n = 1$, there is one couple, and no handshakes can occur, proving the base case. Now suppose the result holds for n couples; we will prove it is valid for $n + 1$ couples. With $n + 1$ couples present, the woman receives $2n + 1$ different answers to her question. But no person P can shake more than $2n$ hands (for $2n + 2$ people, minus P and P's spouse); hence, these $2n + 1$ numbers must be exactly $0, 1, 2, \ldots, 2n$ in some order. In particular, one of these people, A, shook everyone else's hand except A's own spouse (that accounts for the "$2n$" answer), and another, B, shook no hands (the "0" answer). Because B did not shake A's hand, A and B must be married to each other. The remaining $2n$ people include the woman who asked the question, together with those who answered $1, 2, \ldots, 2n - 1$ to her question. Now pretend that A and B had not attended the party, so we are left with n couples. Each of these people shook hands with A and not with B; therefore, when A and B are removed, their handshake counts become $0, 1, 2, \ldots, 2n-2$. Hence, by the induction hypothesis, the questioner shook $n - 1$ hands. But now, if we put A and B back in, we note that the woman shook A's hand as well (and not B's). So, altogether, she shook n hands. This completes the induction step, and now the proof is done.

3. Let $ABCD$ be a square and E a point on side CD. The circle inscribed in triangle ADE touches DE at F, and the circle inside quadrilateral $ABCE$, tangent to sides AB, BC, EA, touches AB at G. Prove that lines AE, BD, and FG meet in a point.

   Solution: Extend lines BC and AE to intersect at H. Then the circle inside quadrilateral $ABCE$, tangent to AB, BC, and EA, is really the inscribed circle of $\triangle HBA$. (Actually, this is only true if the circle lies inside $\triangle HBA$ rather than outside it. However, the fact that CE is parallel to AB with $CE < CD = AB$ readily implies that E lies between H and A, and C lies between H and B, so that the whole quadrilateral $ABCE$ lies within $\triangle HBA$, so the circle drawn inside it does too.) Now let P be the intersection point of lines BD and AE. Consider the homothety (scaling) about P that sends point D to point B. Since homotheties preserve directions of lines, this map takes line AD to the line through B and parallel to AD, namely line HB. Similarly, it takes line DE to line BA. And line EA passes through P, the center of the homothety, so it goes to itself.

   Thus, our homothety takes lines AD, DE, EA to lines $HB, BA, AH (= AE)$, respectively, so it takes $\triangle ADE$ to $\triangle HBA$. Consequently, the incircle of $\triangle ADE$ is mapped to the incircle of $\triangle HBA$, and the map also matches corresponding tangency points: F goes to G. But if a homothety about P takes F to G, then P, F, G must be collinear. We now know that P lies on lines AE, BD, and FG, which is what we need.

4. There are $3,999,999$ cities in Antarctica, and some pairs of them are connected by roads. It is known that, given any two cities, there is a sequence of roads leading from one to the other. Prove that the cities can be divided into 1999 groups (of 2001 cities each) such that, given any two cities in the same group, it is possible to get from one to the other using at most 4000 roads.

Solution: First, we provide some relevant graph-theoretic background. Any finite, connected graph can be turned into a *tree* (a connected graph without cycles) by removing some edges. Proof: If our graph has a cycle, any edge of that cycle can be removed without disconnecting the graph. So remove this edge, leaving a new graph. If it has a cycle, we can again remove an edge; continuing in this manner, we must eventually stop, since there are only finitely many edges to remove. We then have a graph with no cycles; since no edge removal ever disconnected the graph, it must still be connected.

Also, given a tree, we can choose a *root* vertex r. Then, for any vertex v, there is a unique path from v to r, never repeating a vertex (uniqueness follows from the absence of cycles). We call v a *descendant* of w if this path goes through w. Every vertex is considered to be a descendant of itself and of r. Suppose v is a descendant of w; then the path from v to r consists of the path from v to w followed by the path from w to r. It follows that descent is transitive: if w in turn is a descendant of u, then v is a descendant of u. It also follows that $d(v,r) = d(v,w) + d(w,r)$, where $d(x,y)$ denotes the distance (i.e. number of edges in the path) from x to y. Finally, a vertex v with no descendants can be removed and the graph will remain connected. Proof: every other vertex is connected to r by a path that does not pass through v, so these vertices will remain connected to r — and hence to each other — when v is removed.

Now we can solve our original problem. We state the graph-theoretic translation: given a connected graph G on kn vertices ($k \geq 0, n \geq 1$), these vertices can be partitioned into k sets of size n such that $d(v,w) \leq 2n-2$ whenever v,w are in the same set. (In our case, $k = 1999, n = 2001$.) We prove this by induction on k. If $k = 0$, we form no vertex sets, and the statement is vacuously true. Now suppose the statement holds for $k-1$, where $k \geq 1$, and we have a graph G on kn vertices. It suffices to prove the result when G is a tree, since otherwise we can make it a tree by removing some edges and partition the vertices of this tree appropriately. The same partition will then work for the original graph G, since the distance between two vertices cannot increase when we put the deleted edges back.

So suppose G is a tree, and arbitrarily choose a root r. Now let a be a vertex for which $d(a,r)$ is maximal. Let $a = v_1, v_2, \ldots, v_q = r$ be the path from a to r, and choose the smallest i such that v_i has at least n descendants. (Some such i surely exists, since r has kn descendants.) Let S be the set of descendants of v_i; note that if $v \in S$, then every descendant of v is in S, by transitivity. Notice that $v_1, v_2, \ldots, v_{i-1}$ are all descendants of v_{i-1}, so the minimality of i implies $i-1 < n$. Thus, $d(a, v_i) = i-1 \leq n-1$. Now we claim the distance between any two elements of S is at most $2n-2$. Indeed, suppose $b, c \in S$. We have $d(b, v_i) = d(b,r) - d(v_i, r) \leq d(a,r) - d(v_i,r)$ (by choice of a) $= d(a, v_i) \leq n-1$. Similarly, $d(c, v_i) \leq n-1$, and so $d(b,c) \leq d(b,v_i) + d(v_i, c) \leq 2(n-1)$, as claimed.

Now let a_1 be an element of S at maximal distance from r. (For example, take $a_1 = a$.) Then a_1 can have no descendants (except itself) in G, since if b were a descendant of a_1, we would have $d(b,r) = d(a_1, b) + d(a_1, r)$, contradicting maximality. Thus, we can remove a_1 from G to leave a graph G_1, which is still connected — in fact, still a tree with root r. Now a similar argument shows that, if $a_2 \in S \cap G_1$ is chosen to have maximal distance from r, then a_2 can have no descendants in G_1: any descendant would lie in S, by transitivity, and it would also be farther from r than a_2, violating maximality. So, deleting a_2 from G_1 gives another rooted tree, G_2. Then, we can choose $a_3 \in S \cap G_2$ to be maximally distant from r, and so forth. We continue removing vertices in this manner; since S has at least n elements, we can remove n vertices. Thus, we choose distinct vertices $a_1, a_2, \ldots, a_n$, all of which lie in S; this means that any two of these vertices are at distance $\leq 2n-2$ from each other, and the remaining graph, $G - \{a_1, \ldots, a_n\}$, is still a tree. Now using the induction hypothesis, this remaining graph can be partitioned to form the remaining $k-1$ sets of vertices, and the desired partition of G is accomplished.

Remark: In fact, given an arbitrary connected graph with a vertex r selected, we can define v to be a descendant of w whenever $d(v,r) = d(v,w) + d(w,r)$, and the same solution works, without having to assume the graph is a tree. However, the case of a tree helps to motivate the definition.

5. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers with the following property: if S is any nonempty set of positive integers, there exists $s \in S$ with $a_s \leq \gcd(S)$. Prove that $n!$ is divisible by $a_1 a_2 \cdots a_n$ for every positive integer n.

Solution: Fix n. Arrange the integers $a_1, \ldots, a_n$ in nonincreasing order, $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_n}$. We claim there exists a bijective function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $a_{i_k} \mid f(k)$ for each $k = 1, 2, \ldots, n$. To demonstrate this, we construct f inductively. Suppose that $f(k)$ has been defined for all values of k less than some j, and we wish to define $f(j)$. Let d be the greatest common divisor of $i_1, i_2, \ldots, i_j$. By the hypothesis, there exists some $i \in \{i_1, i_2, \ldots, i_j\}$ such that $a_i \leq d$; since $a_{i_j} = \min\{a_{i_1}, a_{i_2}, \ldots, a_{i_j}\}$, we have $a_{i_j} \leq a_i \leq d$. Now, we know of j distinct multiples of d lying in the set $\{1, 2, \ldots, n\}$ (namely, $i_1, i_2, \ldots, i_j$); this many multiples can only exist if $jd \leq n$. Then, the numbers $a_{i_j}, 2a_{i_j}, \ldots, ja_{i_j}$ are also all in $\{1, 2, \ldots, n\}$, since $ja_{i_j} \leq jd \leq n$. At most $j - 1$ of these can have been used up by the previously defined values $f(1), f(2), \ldots, f(j-1)$, so some value is left over; we define $f(j)$ to be such a value. Then $a_{i_j} \mid f(j)$, as required.

So we can recursively define $f(1), f(2), \ldots, f(n)$ by the above method, and our construction ensures that f is injective. Since it maps the finite set $\{1, 2, \ldots, n\}$ to itself, it must actually be bijective. Since $a_{i_k} \mid f(k)$ for each k, we have

$$a_1 a_2 \cdots a_n = a_{i_1} a_{i_2} \cdots a_{i_n} \mid f(1) f(2) \cdots f(n) = 1 \cdot 2 \cdots n = n!.$$

Remark: In fact, $n!$ is the smallest positive integer that necessarily satisfies this condition. Indeed, if p is any prime, then we can define a_n to be the largest power of p dividing n, for each n, and this produces a sequence meeting the condition of the problem statement. Then $a_1 a_2 \cdots a_n$ is the highest power of p dividing $n!$, so taking the least common multiple of these values over all choices of p gives us $n!$.