Berkeley Math Circle 2000-2001
Monthly Contest #6 — Solutions

1. In triangle ABC, let D be the midpoint of side BC. Let E and F be the feet of the perpendiculars to AD from B and C, respectively. Prove that $BE = CF$.

   Solution: We have $\angle DFC = \pi/2 = \angle DEB$. Also, $\angle CDF = \angle BDE$ since they are vertical angles. (It seems possible that E and F could lie on the same side of D, so that $\angle CDF$ and $\angle BDE$ would be supplementary rather than equal; however, if they were supplementary and unequal, then one of them, say $\angle BDE$, would be $> \pi/2$, so the sum of the angles of $\triangle BDE$ would be $> \pi$, a contradiction.) These equal angles imply $\triangle DFC \sim \triangle DEB$. But $CD = BC/2 = BD$, so in fact $\triangle DFC \cong \triangle DEB$, giving $CF = BE$.

   Alternate Solution: (Thanks to Inna Zakharevich) Let H be the foot of the perpendicular from A to BC. Then, using the $bh/2$ formula and the fact that D is the midpoint of BC, we get

   $$\frac{AD \cdot BE}{2} = \text{Area}(\triangle ABD) = \frac{BD \cdot AH}{2} = \frac{CD \cdot AH}{2} = \text{Area}(\triangle ACD) = \frac{AD \cdot CF}{2}.$$

   Multiplying by $2/AD$ now gives $BE = CF$.

2. Let ABC be an equilateral triangle, and let P be a point on minor arc BC of the circumcircle of ABC. Prove that $PA = PB + PC$.

   Solution: Extend line PC through C to point D such that $CD = BP$. Note that $\angle ACD = \pi - \angle PCA = \angle ABP$ (since quadrilateral $ABPC$ is cyclic), and $AC = AB$ since $\triangle ABC$ is equilateral. Consequently, $\triangle ACD \cong \triangle ABP$. In particular, we have $\angle PDA = \angle CDA = \angle BPA = \angle BCA$ (again by cyclicity) $= \pi/3$. But also $\angle APD = \angle APC = \angle ABC$ (cyclicity) $= \pi/3$. We conclude that triangle APD is equilateral. So, $PA = PD = PC + CD = PC + BP$.

   Remark: This is a special case of Ptolemy's Theorem: if $RSTU$ is any convex cyclic quadrilateral, then $RS \cdot TU + ST \cdot RU = RT \cdot SU$. The proof of the theorem is similar to the above.

3. Determine all triples (x, y, n) of integers such that $x^2 + 2y^2 = 2^n$.

   Solution: It is easy to check that $(\pm 2^r, 0, 2r)$ and $(0, \pm 2^r, 2r+1)$ satisfy this equation for any nonnegative integer r. We will show that these are all the solutions by an infinite descent method.

   So suppose we have some solution (x_0, y_0, n_0). If x_0 is odd, then 2^{n_0} is odd, which forces $n_0 = 0$ and then $x_0^2 + 2y_0^2 = 1$, so $y_0 = 0$ (or else $2y_0^2 > 1$ already) and then $x_0 = \pm 1$. On the other hand, if x_0 is even, we can let $x_0 = 2x_0'$ and then $4{x_0'}^2 + 2y_0^2 = 2^{n_0} \Rightarrow y_0^2 + 2{x_0'}^2 = 2^{n_0 - 1}$, so $(x_1, y_1, n_1) = (y_0, x_0/2, n_0 - 1)$ is another solution to our equation, where n_0 has been replaced by $n_0 - 1$. Now if x_1 is even, we can repeat this construction to get another new solution (x_2, y_2, n_2) with $n_0 - 1$ replaced by $n_0 - 2$, and so on. These integers n cannot go on decreasing forever, since there does not exist an integral solution where $n < 0$. Thus, eventually our process terminates, which means we get to a solution (x_k, y_k, n_k) with x_k odd. By the above, this is possible only if $x_k = \pm 1, y_k = 0, n_k = 0$.

   On the other hand, the above construction can be performed in reverse: we have $x_i = 2y_{i+1}, y_i = x_{i+1}, n_i = n_{i+1} + 1$ for each value of $i \geq 0$. Now we claim that $(x_i, y_i, n_i) = (\pm 2^{(k-i)/2}, 0, k-i)$ when $k - i$ is even, and $(0, \pm 2^{(k-i-1)/2}, k-i)$ when $k - i$ is odd. The proof is by downward induction: the base case $i = k$ is certainly true; given that the statement holds for some $i > 0$, it is simple algebra to check that it holds for $i - 1$ by applying our reverse construction. Thus, the claim is true for each $i \geq 0$. In particular, $(x_0, y_0, n_0) = (\pm 2^{k/2}, 0, k)$ or $(0, \pm 2^{(k-1)/2}, k)$, which fits the form above. So, every solution is of this form.

   Remark: For those who like heavy machinery, this problem can also be solved quite rapidly by using unique factorization in the ring $\mathbb{Z}[\sqrt{-2}]$, factoring $x^2 + 2y^2$ as $(x + \sqrt{-2}y)(x - \sqrt{-2}y)$.

4. Suppose that S is a set of 2001 positive integers, and n different subsets of S are chosen so that their sums are pairwise relatively prime. Find the maximum possible value of n. (Here the "sum" of a finite set of numbers means the sum of its elements; the empty set has sum 0.)

Solution: The answer is $2^{2000}+1$. To see that we cannot do better than this, note that at least half of the 2^{2001} possible subsets of S have even sums. Indeed, if all elements of S are even, then all subsets have even sums; on the other hand, if there exists some odd $s \in S$, we can divide the subsets of S into pairs of the form $\{T, T \cup \{s\}\}$ for each subset T not containing s. Since the sum of T and that of $T \cup \{s\}$ are of opposite parity, each pair contains exactly 1 subset with an even sum. So, in this case, half the subsets of S have even sums. The upshot is that, in either case, there are at most 2^{2000} subsets of S with odd sums. Since our chosen subsets can include at most one subset whose sum is even (because no two sums can have a common factor of 2), we cannot choose more than $2^{2000}+1$ subsets altogether.

Now, we must construct an example to show that we can have $n = 2^{2000}+1$. To do this, let $k = (2^{2000})!$, and let $S = \{k, 2k, 4k, 8k, \ldots, 2^{1999}k, 1\}$. We consider the 2^{2000} subsets containing the element 1, plus the one subset $\{k\}$. It is evident that k, the sum of the last subset, is relatively prime to the sum of any subset containing 1, since this latter sum is of the form $ak+1$ for some a. So now we just need to prove that any two distinct subsets containing 1 have relatively prime sums. Well, any such set consists of several distinct powers of 2, multiplied by k, plus 1. The sum of these powers of 2 is some number $a, 0 \le a < 2^{2000}$. Thus the subset's sum is $ak+1$. However, it follows from the uniqueness of binary representation that, for each possible value of a, there is only one subset whose sum is $ak+1$. Consequently, if we choose another, different subset (also containing 1), its sum is $bk+1$ for some $b, 0 \le b < 2^{2000}$ with $a \neq b$. Now suppose $ak+1$ and $bk+1$ are not relatively prime; then they have some common prime factor p. So $p \mid ak+1$ and $p \mid bk+1$, hence $p \mid (ak+1)-(bk+1) = (a-b)k$. Then, $p \mid a-b$ or $p \mid k$. But $a-b$ is nonzero and has absolute value $< 2^{2000}$, so $a-b$ is one of the factors in the product $1 \cdot 2 \cdot 3 \cdots 2^{2000} = k$, and we get $a-b \mid k$. Thus, we are guaranteed that p divides k. But then p cannot divide $ak+1$, so we have a contradiction. We conclude that our subset sums are, in fact, pairwise relatively prime, completing the proof.

5. Let $x_1, x_2, \ldots, x_{1000}, y_1, y_2, \ldots, y_{1000}$ be 2000 different real numbers, and form the 1000×1000 matrix whose (i,j)-entry is $x_i + y_j$. If the product of the numbers in each row is 1, show that the product of the numbers in each column is -1.

Solution: The given says that $(x_i+y_1)(x_i+y_2)\cdots(x_i+y_{1000}) = 1$ for each $i = 1, 2, \ldots, 1000$. So, if we let $P(x)$ be the polynomial $(x+y_1)(x+y_2)\cdots(x+y_{1000})-1$, the numbers x_i are all roots of P. These numbers are all distinct, and there are 1000 of them. But P, being of degree 1000, can only have 1000 roots, so these are all the roots of P and the polynomial factors as $P(x) = c(x-x_1)(x-x_2)\cdots(x-x_{1000})$ for some constant c. Since the leading coefficient of P is 1, we conclude that $c = 1$. Thus,

$$(x+y_1)(x+y_2)\cdots(x+y_{1000}) - 1 = (x-x_1)(x-x_2)\cdots(x-x_{1000})$$

is a polynomial identity, valid for all x.

Now choose any j $(1 \le j \le 1000)$; we wish to show that the product of the numbers in the jth column of the matrix is -1. Letting $x = -y_j$ in the above equation, we get

$$\begin{aligned}(-y_j+y_1)(-y_j+y_2)\cdots(-y_j+y_{1000}) - 1 &= (-y_j-x_1)(-y_j-x_2)\cdots(-y_j-x_{1000}) \\ &= (-1)^{1000}(x_1+y_j)(x_2+y_j)\cdots(x_{1000}+y_j).\end{aligned}$$

However, the product on the left-hand side is 0, since the jth factor is $-y_j+y_j = 0$; also, $(-1)^{1000} = 1$. Thus, we obtain $-1 = (x_1+y_j)(x_2+y_j)\cdots(x_{1000}+y_j)$, which is what we wanted to prove.