Berkeley Math Circle 2000-2001
Monthly Contest #5 — Solutions

1. 2000 distinct positive integers are written down, and it so happens that the product of any 3 different numbers from this list is a square. Prove that each one of them is a square.

   Solution: Let a be any number in the list; we wish to prove that a is a square. Let b, c, d be any three other elements of the list, all different from each other. Then, the given implies that there are integers p, q, r with $abc = p^2, abd = q^2, acd = r^2$. Multiplying these together gives $a^3b^2c^2d^2 = p^2q^2r^2$; equivalently, $a = (p^2q^2r^2)/(a^2b^2c^2d^2) = (pqr/abcd)^2$. So a is the square of a rational number, but since it is an integer, we conclude that it is the square of an integer, as needed.

2. Martha is a chicken who lives in a coop whose finitely many residents have a definite "pecking order": for every pair of distinct chickens, exactly one pecks the other. (However, it is not necessarily true that if X pecks Y and Y pecks Z, then X pecks Z.) A chicken X is called a "leader" of the coop if every other chicken is pecked by X or pecked by a chicken who is pecked by X. Prove that if no one in the coop pecks more chickens than Martha, then Martha is a leader.

   Solution: Suppose that no one pecks more chickens than M, yet M is not a leader; we will seek a contradiction. Let $X_1, X_2, \ldots, X_n$ be the chickens pecked by M. Since M is not a leader, there is a chicken N who is neither pecked by M nor by any of the X_i. But it is given that of any two distinct chickens, one pecks the other; hence, N pecks all of $M, X_1, X_2, \ldots, X_n$. Thus, we have found at least $n+1$ chickens pecked by N. But M only pecks n chickens. Since we assumed nobody pecked more chickens than M, we have a contradiction, as needed.

3. If a, b, c, d, e, f are real numbers such that $a+b+c+d+e+f = 0$ and $a^3+b^3+c^3+d^3+e^3+f^3 = 0$, prove that $(a+c)(a+d)(a+e)(a+f) = (b+c)(b+d)(b+e)(b+f)$.

   Solution: First, let S be the sum of all possible products of three distinct variables from the set $\{a, b, c, d, e, f\}$, i.e. $S = abc + abd + abe + \cdots + def$. Now, we assert that

$$(a+b+c+d+e+f)^3 + 2(a^3+b^3+c^3+d^3+e^3+f^3) -$$
$$3(a+b+c+d+e+f)(a^2+b^2+c^2+d^2+e^2+f^2) = 6S. \quad (1)$$

   To see this, note that, when the left-hand side is expanded, the terms obtained come in three forms: a^3 (i.e. cubes of one variable), a^2b (squares of one variable times another), and abc (products of three different variables). Each term of the form a^3 has coefficient 1 in $(a+\cdots+f)^3$, 2 in $2(a^3+\cdots+f^3)$, and -3 in $-3(a+\cdots+f)(a^2+\cdots+f^2)$, for a total coefficient of 0; the terms of the form a^2b have respective coefficients $3, 0, -3$, for a total of 0; those of the form abc have coefficients $6, 0, 0$, for a total of 6. Since the right side of (??) consists of the abc-type terms with coefficient 6, the assertion is proven.

   However, the given tells us that the left-hand side of (??) is zero, so $S = 0$. Now, let $p = ab - (cd + ce + cf + de + df + ef)$. We claim that the following holds for all numbers x:

$$(x+c)(x+d)(x+e)(x+f) = (x-a)(x-b)(x^2-p) + abp + cdef. \quad (2)$$

   We can verify this by comparing coefficients for corresponding powers of x.

   The coefficient of x^4 on both sides is 1; the coefficient of x^3 is $c+d+e+f$ on the left and $-(a+b)$ on the right, but the given implies that these are equal. The coefficient of x^2 is $cd+ce+cf+de+df+ef$ on the left and $ab-p$ on the right; these are equal by the definition of p. The coefficient of x is $cde+cdf+cef+def$ on the left and $ap+bp$ on the right. To see that these are equal, observe that

$$cde+cdf+cef+def-(a+b)p = cde+cdf+cef+def-(a+b)ab+(a+b)(cd+ce+cf+de+df+ef)$$
$$= (S-[abc+abd+abe+abf]) - (a+b)ab = -ab(c+d+e+f+a+b) = 0,$$

where we have used the definition of S for the second equality and the fact that $S = 0$ for the third. Finally, the constant term equals $cdef$ on both sides. Thus, equation (??) holds.

Plugging in $x = a$ and $x = b$, we find

$$(a+c)(a+d)(a+e)(a+f) = (a-a)(a-b)(a^2-p) + abp + cdef = abp + cdef,$$

$$(b+c)(b+d)(b+e)(b+f) = (b-a)(b-b)(b^2-p) + abp + cdef = abp + cdef.$$

Since the right-hand sides match, we are done.

4. Let $ABCD$ be a cyclic quadrilateral (i.e. inscribed in a circle) such that $DC = AD + BC$. Prove that the intersection point of the angle bisectors of angle A and angle B lies on CD.

   Solution: Let E lie on segment CD so that $CE = BC$. Then, $DE = DC - CE = AD$. So, $\triangle BCE$ and $\triangle ADE$ are both isosceles. Let $\alpha = \angle DAE = \angle AED$, and let $\beta = \angle CEB = \angle EBC$. Then $\angle CDA = \angle EDA = \pi - 2\alpha$, but since the quadrilateral is cyclic, this gives $\angle ABC = 2\alpha$. Likewise, $\angle BCD = \pi - 2\beta$ and $\angle DAB = 2\beta$.

   Now, let the circumcircle of triangle ABE intersect line CD again at F. We will assume that F lies on the same side of E as D (the other case is completely analogous). Since quadrilateral $ABEF$ is cyclic, we have $\angle ABF = \angle AEF = \alpha = \angle ABC/2$, and $\angle FAB = \pi - \angle BEF = \angle CEB = \beta = \angle DAB/2$. So F lies on the bisectors of angles DAB and ABC, which means it is precisely the intersection point specified in the problem statement. Since F was constructed to lie on CD, we are done.

5. In the UC Berkeley bureaucratic hierarchy, certain administrators report to certain other administrators. It so happens that if A reports to B and B reports to C, then C reports to A. Also, administrators do not report to themselves. Prove that all the administrators can be divided into three disjoint groups X, Y, Z so that, if A reports to B, then either A is in X and B is in Y, or A is in Y and B is in Z, or A is in Z and B is in X.

   Solution: Form a graph whose vertices represent the administrators, with an edge between A and B if one of A, B reports to the other. We will temporarily assume that the graph is connected. Consider any walk on this graph (i.e. a finite sequence of vertices, any two consecutive members of which are connected by an edge). Whenever this walk traverses an edge from A to B, assign this edge a weight of 1 if A reports to B and -1 if B reports to A; then let the "value" of the walk be the sum of the weights of its edges. This definition is unambiguous, i.e. we cannot have both A reporting to B and B reporting to A, since then we would also have to have A reporting to A, which is prohibited.

   We claim that if a walk is a cycle (it starts and ends at the same vertex), its value is divisible by 3. If not, choose a counterexample that uses as few edges as possible. Let its value be v. We consider the possible weights of successive edges in our cycle. First, suppose there are two consecutive 1's, representing some movements from A to B and from B to C. So A reports to B, and B reports to C; hence, from the given, C reports to A, and we can replace these two edges ABC by one edge AC, whose weight will be -1. So the new cycle we have obtained has one edge less than the old one, and its value is $v-3$, which is not divisible by 3 (since v wasn't). This contradicts the minimality of the original cycle, so this situation is impossible. Similarly, if there were two consecutive -1's, corresponding to edges CBA, we could replace them with an edge CA of weight 1, and we would have a new cycle, shorter than the previous one, of value $v+3$, which again is not divisible by 3. So this, too, is impossible.

   We conclude that the edge weights of our cycle must alternate between 1 and -1. The initial and final edges must also have opposite weights, or else we can replace them with one edge to obtain a shorter counterexample, precisely as in the previous paragraph. Actually, we avoid this problem if the initial and final edges are the same, so that our minimal cycle consists of only one edge, but then this edge runs from A to A, which is illegal. So, the only remaining possibility is that our cycle weights are of the form $1, -1, 1, -1, \ldots, -1$, or $-1, 1, -1, 1, \ldots, 1$, but either way, the sum is 0, which is in fact a multiple of 3. So we have ruled out all the cases and found that a counterexample to our claim is impossible. Thus, the claim holds.

   Now choose any vertex P. For Q any other vertex, we can define the value of Q to be the element of the set $\{0, 1, 2\}$ which is congruent mod 3 to the value of a walk from P to Q. (Such a walk exists, since the graph is connected). This is well-defined as long as we know that the value of Q is independent of the walk chosen. But if there exist two walks from P to Q with respective values v, w, then traversing the first walk from P to Q and traversing the second walk backwards from Q to P gives a cycle with value $v - w$. The above claim then shows that $v - w$ is divisible by 3, so both walks do give the same value, as required.

Now just let X consist of all vertices of value 0 (including P), Y all vertices of value 1, and Z those of value 2. If AB is an edge (with A reporting to B), then it is clear that a walk from P to A, followed by the edge AB, gives a walk from P to B. So, the value of B equals the value of A plus 1 (mod 3). Consequently: if $A \in X$ then $B \in Y$; if $A \in Y$ then $B \in Z$; if $A \in Z$ then $B \in X$.

Thus, the problem is solved in the connected case. But if our administrator graph was disconnected, let the components be C_i for some set of indices i. For each C_i, partition it into sets X_i, Y_i, Z_i by the connected case; then, let $X = \cup X_i, Y = \cup Y_i, Z = \cup Z_i$. Since any two administrators connected by an edge must lie in the same C_i, it easily follows that these sets X, Y, Z do the trick.

Remark: Work submitted by Neil Herriot indicated an alternate solution worth mentioning, though difficult to present in full rigor: the administrators can be successively numbered and assigned to sets X, Y, Z in such a manner that, if there is an "inconsistency" (the requirements of the problem statement are not fulfilled) between two administrators, there must also be an inconsistency between two administrators with smaller numbers, leading to an infinite descent and contradiction. Although the wording of the problem suggests that there are finitely many administrators, the solution presented here works if there are infinitely (even uncountably) many, an advantage over the inductive solution.