Berkeley Math Circle Monthly Contest #5
Due February 4, 2000

1. 2000 distinct positive integers are written down, and it so happens that the product of any 3 different numbers from this list is a square. Prove that each one of them is a square.

2. Martha is a chicken who lives in a coop whose finitely many residents have a definite "pecking order": for every pair of distinct chickens, exactly one pecks the other. (However, it is not necessarily true that if X pecks Y and Y pecks Z, then X pecks Z.) A chicken X is called a "leader" of the coop if every other chicken is pecked by X or pecked by a chicken who is pecked by X. Prove that if no one in the coop pecks more chickens than Martha, then Martha is a leader.

3. If a, b, c, d, e, f are real numbers such that $a + b + c + d + e + f = 0$ and $a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = 0$, prove that $(a + c)(a + d)(a + e)(a + f) = (b + c)(b + d)(b + e)(b + f)$.

4. Let $ABCD$ be a cyclic quadrilateral (i.e. inscribed in a circle) such that $DC = AD + BC$. Prove that the intersection point of the angle bisectors of angle A and angle B lies on CD.

5. In the UC Berkeley bureaucratic hierarchy, certain administrators report to certain other administrators. It so happens that if A reports to B and B reports to C, then C reports to A. Also, administrators do not report to themselves. Prove that all the administrators can be divided into three disjoint groups X, Y, Z so that, if A reports to B, then either A is in X and B is in Y, or A is in Y and B is in Z, or A is in Z and B is in X.

Please write solutions to different problems on separate pages. At the top of each page, write your name, school, city, contest number, problem number, and the division in which you are participating (beginner or advanced). Please go to **http://mathcircle.berkeley.edu** for more information about the contest, or email questions to gastropodc@hotmail.com or andrew@dudzik.com. ©Berkeley Math Circle