Berkeley Math Circle Monthly Contest #5
Due February 4, 2000

1. 2000 distinct positive integers are written down, and it so happens that the product of any 3 different numbers from this list is a square. Prove that each one of them is a square.

2. Martha is a chicken who lives in a coop whose finitely many residents have a definite "pecking order": for every pair of distinct chickens, exactly one pecks the other. (However, it is not necessarily true that if X pecks Y and Y pecks Z, then X pecks Z.) A chicken X is called a "leader" of the coop if every other chicken is pecked by X or pecked by a chicken who is pecked by X. Prove that if no one in the coop pecks more chickens than Martha, then Martha is a leader.

3. If a, b, c, d, e, f are real numbers such that $a + b + c + d + e + f = 0$ and $a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = 0$, prove that $(a + c)(a + d)(a + e)(a + f) = (b + c)(b + d)(b + e)(b + f)$.

4. Let $ABCD$ be a cyclic quadrilateral (i.e. inscribed in a circle) such that $DC = AD + BC$. Prove that the intersection point of the angle bisectors of angle A and angle B lies on CD.

5. In the UC Berkeley bureaucratic hierarchy, certain administrators report to certain other administrators. It so happens that if A reports to B and B reports to C, then C reports to A. Also, administrators do not report to themselves. Prove that all the administrators can be divided into three disjoint groups X, Y, Z so that, if A reports to B, then either A is in X and B is in Y, or A is in Y and B is in Z, or A is in Z and B is in X.