Inequalities and Triangles

Tom Rike

Berkeley Math Circle

December 5, 1999

1 Euler's Inequality

One of the oldest inequalities about triangles is that relating the radii of the circumcircle and incircle. It was proved by Euler and is contained in the following theorems. Proofs are given in *Geometry Revisited* by Coxeter and Greitzer. It is published by the Mathematical Association of America and should be on the bookshelf of everyone interested in geometry.

Theorem 1 (Euler 1765) *Let O and I be the circumcenter and incenter, respectively, of a triangle with circumradius R and inradius r; let d be the distance OI. Then*

$$d^2 = R^2 - 2Rr$$

Theorem 2 *In a triangle with circumradius R and inradius r, $R \geq 2r$.*

Here are seven other interesting and useful facts about triangles. Let s denote the semiperimeter of triangle ABC, α, β, γ the angles, a, b, c the opposite sides, and K the area.

1. $K = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha$.
2. $K = \sqrt{s(s-a)(s-b)(s-c)}$. (Heron's formula)
3. $K = rs$
4. $2R = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$ (Law of Sines).
5. $K = \frac{abc}{4R}$.
6. $1 + \cos\alpha = \frac{(a+b+c)(-a+b+c)}{2bc}$ $\qquad 1 - \cos\alpha = \frac{(a-b+c)(a+b-c)}{2bc}$.
7. $\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ $\qquad \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$ $\qquad \tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{r}{s-a}$.

Formulas similar to those in (6) and (7) can also be written for the angles β and γ. To see (1), drop an altitude from C to c forming a right triangle. The area is one-half the product of the base c and the altitude. But the altitude equals $a\sin\beta$. To see (2), again drop an altitude, h, forming two right triangles with bases x and $c-x$. Use the Pythagorean Theorem twice and eliminate the altitude to solve for $x = \frac{a^2-b^2+c^2}{2c}$ (Note $x = a\cos\beta$). Now, substitute x back into $h^2 = a^2 - x^2$. Use $A^2 - B^2 = (A-B)(A+B)$ and $A^2 + 2AB + B^2 = (A+B)^2$ to expand. Then multiply by $4c^2$ giving $(b+c-a)(a+b-c)(a+c-b)(a+b+c)$. For more details see pages

337-338 of *Geometry*, Second Edition by Harold Jacobs. For a proof using trigonometry see *Cyclic quadrangles; Brahmagupta's formula* on pages 56-59 of *Geometry Revisited* by Coxeter and Greitzer. Heron's formula is then seen to be a corollary to Brahmagupta's formula. To see (3), divide the triangle into three triangles with segments from the incenter to the vertices. To see (4), circumscribe the triangle and draw a diameter from one of the vertices. Draw a chord from the other endpoint of the diameter to a second vertex of the triangle. Note that the angle at the third vertex is equal to the angle formed by the diameter and the chord, or supplementary to it, if the third angle is not acute. Therefore, the two angles have equal sines. To see (5), use (1) and (4). To see (6), solve the Law of Cosines for $\cos\alpha$ and add 1 or subtract from 1. To see (7), use the half-angle formulas $\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$, $\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$, and (6). For the final part of (7) use the first two parts of (7) and formulas (2) and (3).

2 Convex Functions and Jensen's Inequality

A real-valued function f is *convex* on an interval I if and only if

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b) \tag{1}$$

for all $a, b \in I$ and $0 \le t \le 1$. This just says that a function is convex if the graph of the function lies below its secants. See pages 2 through 5 of Bjorn Poonen's paper, distributed at his talk on inequalities, for a discussion of convex functions and inequalities for convex functions. A number of common functions that are convex are also listed. Among those listed are $-\ln x$ on $(0,\infty)$, $-\sin x$ on $[0,\pi]$, $-\cos x$ on $[-\pi/2, \pi/2]$ and $\tan x$ on $[0, \pi/2]$. To avoid the negative signs a complementary concept is defined. A real-valued function f is *concave* on an interval I if and only if

$$f(ta + (1-t)b) \ge tf(a) + (1-t)f(b) \tag{2}$$

for all $a, b \in I$ and $0 \le t \le 1$. Therefore f is convex iff $-f$ is concave. If you are familiar with derivatives then the following theorem about twice differentiable functions provides a way of telling if such a function is convex.

Theorem 3 *If $f''(x) \ge 0$ for all $x \in I$, then f is convex on I.*

Inequality (**1**) can be generalized to a convex function f with three variables x_1, x_2, x_3 with weights t_1, t_2, t_3, respectively, such that $t_1 + t_2 + t_3 = 1$. Note that $t_2 + t_3 = 1 - t_1$. In this manner the three variable case can be transformed into the two variable case as follows.

$$\begin{aligned}
f(t_1x_1 + t_2x_2 + t_3x_3) &= f\left(t_1x_1 + (1-t_1)\frac{t_2x_2 + t_3x_3}{t_2 + t_3}\right) \\
&\le t_1f(x_1) + (1-t_1)f\left(\frac{t_2x_2 + t_3x_3}{t_2 + t_3}\right) \\
&= t_1f(x_1) + (1-t_1)f\left(\frac{t_2}{t_2 + t_3}x_2 + \frac{t_3}{t_2 + t_3}x_3\right) \\
&\le t_1f(x_1) + (t_2 + t_3)\left(\frac{t_2}{t_2 + t_3}f(x_2) + \frac{t_3}{t_2 + t_3}f(x_3)\right) \\
&= t_1f(x_1) + t_2f(x_2) + t_3f(x_3).
\end{aligned}$$

This process can be continued to produce an n variable version which is due to J.L.W.V. Jensen. It can be easily proved by mathematical induction using the above technique. Write your own proof and compare with the one given here. It will give you some good practice manipulating sigma notation.

Theorem 4 (Jensen's Inequality 1906) *Let f be a convex function on the interval I. If $x_1, x_2, \ldots, a_n \in I$ and $t_1, t_2, \ldots, t_n$ are nonnegative real numbers such that $t_1 + t_2 + \ldots + t_n = 1$, then*

$$f(\sum_{i=1}^{n} t_i x_i) \leq \sum_{i=1}^{n} t_i f(x_i).$$

Proof by induction: The case for $n = 2$ is true by the definition of convex. Assume the relation holds for n, then we have

$$\begin{aligned}
f\left(\sum_{i=1}^{n+1} t_i x_i\right) &= f\left(\sum_{i=1}^{n} t_i x_i + t_{n+1}x_{n+1}\right) = f\left(t_{n+1}x_{n+1} + (1-t_{n+1})\frac{1}{1-t_{n+1}}\sum_{i=1}^{n} t_i x_i\right) \\
&\leq t_{n+1}f(x_{n+1}) + (1-t_{n+1}) f\left(\frac{1}{1-t_{n+1}}\sum_{i=1}^{n} t_i x_i\right) \\
&= t_{n+1}f(x_{n+1}) + (1-t_{n+1}) f\left(\sum_{i=1}^{n} \frac{t_i}{1-t_{n+1}} x_i\right) \\
&\leq t_{n+1}f(x_{n+1}) + (1-t_{n+1}) \sum_{i=1}^{n} \frac{t_i}{1-t_{n+1}} f(x_i) \\
&= \sum_{i=1}^{n} t_i f(x_i) + t_{n+1}f(x_{n+1}) \\
&= \sum_{i=1}^{n+1} t_i f(x_i).
\end{aligned}$$

Thus showing that the assumption implies that the relation holds for $n+1$ and by the principle of Mathematical Induction holds for all natural numbers.

■

An easy consequence of Jensen's theorem is the following proof of the arithmetic mean-geometric mean inequality. (Problem 13 from Bjorn's paper)

Theorem 5 (AM-GM Inequality) *If $x_1, x_2, \ldots, x_n \geq 0$ then*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Proof. Since $-\ln x$ is convex then $\ln x$ is concave. By Jensen's theorem we have

$$\begin{aligned}
\ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) &\geq \frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n} \\
&= \frac{1}{n}\ln(x_1 x_2 \cdots x_n) \\
&= \ln[(x_1 x_2 \cdots x_n)^{\frac{1}{n}}]
\end{aligned}$$

Since $\ln x$ is monotonic increasing ($f'(x) = \frac{1}{x} > 0$) for $x > 0$ we have
$\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$.

■

The proof of Jensen's Inequality does not address the specification of the cases of equality. It can be shown that strict inequality exists unless all of the x_i are equal or f is linear on an interval containing all of the x_i.

3 Seven Wonders of the World

The title of this section comes from an article by Richard Hoshino entitled *The Other Side of Inequalities, Part Four* in *Mathematical Mayhem*, Volume 7, issue 4, March-April 1995. *Mathematical Mayhem* merged with *Crux Mathematicorum* after Volume 8 and the two are published together by the Canadian Mathematical Society eight times a year. The cost for nonmembers is $60 a year, but a student rate of only $20 available. The article was a very successful attempt to show how some inequalities could be elegantly solved with some trigonometry. The major theme was to employ Jensen's Inequality for *concave* functions of three variables. Namely,

$$\frac{f(x_1)+f(x_2)+f(x_3)}{3} \le f\left(\frac{x_1+x_2+x_3}{3}\right). \tag{3}$$

Note that $f(x) = \sin x$ is concave on $[0,\pi]$, $f(x) = \csc x$ is convex on $(0,\pi)$, $f(x) = \cos x$ is concave on $[0,\pi/2]$ and convex on $[\pi/2,\pi]$ and $\tan x$ is convex on $(0,\pi/2)$. As before α,β, and γ are the angles of triangle ABC. The following list of inequalities comprise the Seven Wonders of the World.

W1 $\sin\alpha+\sin\beta+\sin\gamma \le \frac{3\sqrt{3}}{2}$.

W2 $\csc\alpha+\csc\beta+\csc\gamma \ge 2\sqrt{3}$

W3 $1 < \cos\alpha+\cos\beta+\cos\gamma \le \frac{3}{2}$.

W4 $\cot\alpha\cot\beta\cot\gamma \le \frac{\sqrt{3}}{9}$.

W5 $\cot\alpha+\cot\beta+\cot\gamma \ge \sqrt{3}$.

W6 $\sin^2\alpha+\sin^2\beta+\sin^2\gamma \le \frac{9}{4}$.

W7 $\cot^2\alpha+\cot^2\beta+\cot^2\gamma \ge 1$.

The following are some proofs that exhibit the usefulness of Jensen's Inequality and some other standard techniques with trigonometric functions.

W1 Since $\sin x$ is concave on $(0,\pi)$ by Jensen's Inequality we have $\frac{\sin\alpha+\sin\beta+\sin\gamma}{3} \le \sin(\frac{\alpha+\beta+\gamma}{3})$. But $\alpha+\beta+\gamma=\pi$, so $\frac{\alpha+\beta+\gamma}{3} = \frac{\pi}{3}$. Multiplying both sides of the inequality by 3 and using $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ gives the result. ■

W2 Since $\csc x$ is convex on $(0,\pi)$ by Jensen's Inequality we have
$\csc\alpha+\csc\beta+\csc\gamma \ge 3\csc[(\alpha+\beta+\gamma)/3] = 3\csc\frac{\pi}{3} = 2\sqrt{3}$. ■

W3 If $\alpha,\beta,\gamma < \frac{\pi}{2}$ then by Jensen's Inequality we have
$\cos\alpha+\cos\beta+\cos\gamma \le 3\cos[(\alpha+\cos\beta+\cos\gamma)/3)] = \frac{3}{2}$. Otherwise the situation becomes complicated. See Richard Hoshino's article for details. For an alternate proof see **Some Harder Problems**, number 3, at the end.

W4 If one of the angles, α, is not acute then the value for $\cot\alpha < 0$ and the values for the other two angles will by positive so that the inequality is clearly true. If the three angles are acute, since $\tan x$ is convex and $\gamma = \pi-(\alpha+\beta)$, we have by Jensen's Inequality $\tan\alpha+\tan\beta+\tan\gamma \ge 3\tan[(\alpha+\beta+\gamma)/3] = 3\sqrt{3}$. But $\tan\alpha+\tan\beta+\tan\gamma = \tan\alpha\tan\beta\tan\gamma$ (Prove this). Therefore $\tan\alpha\tan\beta\tan\gamma \ge 3\sqrt{3}$. Taking the reciprocals we have $\cot\alpha\cot\beta\cot\gamma \le \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}$. ■

W5 First note that $\cot\alpha + \cot\beta = \frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta} = \frac{\sin\beta\cos\alpha + \cos\beta\sin\alpha}{\sin\alpha\sin\beta} = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$.
But

$$\begin{aligned} \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta &\leq 1 \\ -\cos(\alpha+\beta) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta &= \cos\gamma \end{aligned}$$

Adding we get

$$\begin{aligned} 2\sin\alpha\sin\beta &\leq 1 + \cos\gamma \\ 2\sin\alpha\sin\beta\sin(\alpha+\beta) &\leq (1+\cos\gamma)\sin(\alpha+\beta) \\ 2\sin\alpha\sin\beta\sin(\gamma) &\leq (1+\cos\gamma)\sin(\alpha+\beta) \\ \frac{2\sin\alpha\sin\beta\sin(\gamma)}{\sin\alpha\sin\beta(1+\cos\gamma)} &\leq \frac{(1+\cos\gamma)\sin(\alpha+\beta)}{\sin\alpha\sin\beta(1+\cos\gamma)} \\ \frac{2\sin\gamma}{1+\cos\gamma} &\leq \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} \end{aligned}$$

Therefore

$$\begin{aligned} \cot\alpha + \cot\beta + \cot\gamma &= \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} + \cot\gamma \\ &\geq \frac{2\sin\gamma}{1+\cos\gamma} + \frac{\cos\gamma}{\sin\gamma} \\ &= \frac{1}{2}\left(\frac{4\sin^2\gamma + 2\cos^2\gamma + 2\cos\gamma}{(1+\cos\gamma)\sin\gamma}\right) \\ &= \frac{1}{2}\left(\frac{3\sin^2\gamma + \cos^2\gamma + 2\cos\gamma + 1}{(1+\cos\gamma)\sin\gamma}\right) \\ &= \frac{1}{2}\left(\frac{3\sin^2\gamma + (\cos\gamma+1)^2}{(\cos\gamma+1)\sin\gamma}\right) \\ &= \frac{1}{2}\left(\frac{3\sin\gamma}{(\cos\gamma+1)} + \frac{\cos\gamma+1}{\sin\gamma}\right) \\ &\geq \frac{2}{2}\left(\sqrt{\frac{3\sin\gamma}{(\cos\gamma+1)}\frac{\cos\gamma+1}{\sin\gamma}}\right) \quad \textit{By the AM-GM Inequality} \\ &= \sqrt{3} \end{aligned}$$

So $\cot\alpha + \cot\beta + \cot\gamma \geq \sqrt{3}$ ■

W6 Since $\gamma = \pi - (\alpha+\beta)$ and the sine of an angle equals the sine of its supplement we have $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \sin^2\alpha + \sin^2\beta + \sin^2(\alpha+\beta)$

$$\begin{aligned} &= \sin^2\alpha + \sin^2\beta + \sin^2\alpha\cos^2\beta + 2\sin\alpha\sin\beta\cos\alpha\cos\beta + \cos^2\alpha\sin^2\beta \\ &= \sin^2\alpha + \sin^2\beta + (1-\cos^2\alpha)\cos^2\beta + 2\sin\alpha\sin\beta\cos\alpha\cos\beta + \cos^2\alpha(1-\cos^2\beta) \\ &= \sin^2\alpha + \sin^2\beta + \cos^2\beta - \cos^2\alpha\cos^2\beta + 2\sin\alpha\sin\beta\cos\alpha\cos\beta + \cos^2\alpha - \cos^2\alpha\cos^2\beta \\ &= 2 - 2\cos^2\alpha\cos^2\beta + 2\sin\alpha\sin\beta\cos\alpha\cos\beta \\ &= 2 - 2\cos\alpha\cos\beta(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ &= 2 - 2\cos\alpha\cos\beta\cos(\alpha+\beta) \\ &= 2 + 2\cos\alpha\cos\beta\cos(\gamma) \end{aligned}$$

But from W3 we have $\frac{\cos\alpha+\cos\beta+\cos\gamma}{3} \le \frac{1}{2}$ so that $\left(\frac{\cos\alpha+\cos\beta+\cos\gamma}{3}\right)^3 \le \frac{1}{8}$.
By the AM-GM we have $\cos\alpha\cos\beta\cos\gamma \le \left(\frac{\cos\alpha+\cos\beta+\cos\gamma}{3}\right)^3 \le \frac{1}{8}$.
Therefore $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2 + 2\cos\alpha\cos\beta\cos\gamma \le 2 + 2(\frac{1}{8}) = \frac{9}{4}$. ■

W7 By the AM-GM we have $\cot^2\alpha + \cot^2\beta \ge 2\cot\alpha\cot\beta$ and likewise for the other pairs. Adding the three inequalities together and dividing by 2 we have

$$\begin{aligned}\cot^2\alpha + \cot^2\beta + \cot^2\gamma &\ge \cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha\\ &= \cot\alpha\cot\beta - \cot\beta\cot(\alpha+\beta) - \cot(\alpha+\beta)\cot\alpha\\ &= \cot\alpha\cot\beta - \cot(\alpha+\beta)(\cot\beta + \cot\alpha)\\ &= \cot\alpha\cot\beta - \frac{\cot\alpha\cos\beta - 1}{\cot\alpha + \cot\beta}(\cot\beta + \cot\alpha)\\ &= \cot\alpha\cot\beta - \cot\alpha\cos\beta + 1\\ &= 1.\end{aligned}$$

Therefore $\cot^2\alpha + \cot^2\beta + \cot^2\gamma \ge 1$. ■

4 Problems

Now for some exercises upon which to practice these ideas. The first three are easy if you apply the correct trigonometric identity. The next eleven problems apply the Seven Wonders of the World, Jensen's Inequality, AM-GM Inequality and/or previous exercises.

1. If $a^2+b^2 = 1$ and $m^2+n^2 = 1$ for real numbers a, b, m and n, prove that $|am+bn| \le 1$.
2. Solve $3\sin^2\alpha - 4\sin^4\alpha - 2 = 0$.
3. (**1984 ARML**) In triangle ABC, $a \ge b \ge c$. If $\frac{a^3+b^3+c^3}{\sin^3\alpha+\sin^3\beta+\sin^3\gamma} = 7$, compute the maximum possible value for a.
4. $\sin\alpha\sin\beta\sin\gamma \le \frac{3\sqrt{3}}{8}$.
5. $\csc\alpha\csc\beta\csc\gamma \ge \frac{8\sqrt{3}}{9}$.
6. $\frac{3}{4} \le \cos^2\alpha + \cos^2\beta + \cos^2\gamma < 3$.
7. $\sec^2\alpha + \sec^2\beta + \sec^2\gamma > 3$.
8. $\csc^2\alpha + \csc^2\beta + \csc^2\gamma \ge 4$.
9. $1 < \sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \le \frac{3}{2}$.
10. $2 < \cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \le \frac{3\sqrt{3}}{2}$.
11. $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \ge \sqrt{3}$.
12. $\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} \ge 3\sqrt{3}$.
13. $\csc\frac{\alpha}{2} + \csc\frac{\beta}{2} + \csc\frac{\gamma}{2} \ge 6$.
14. $\sec\frac{\alpha}{2} + \sec\frac{\beta}{2} + \sec\frac{\gamma}{2} \ge 2\sqrt{3}$.

5 Hints

1. Use the fundamental trigonometric identity relating $\sin\theta$ and $\cos\theta$.
2. Recall or derive the formula for $\sin 3\theta$ in terms of $\sin\theta$.
3. Use Law of Sines.
4. Use W1 and AM-GM inequality.
5. Use the previous exercise.
6. Use W6.
7. Consider the range of $|\sec\theta|$
8. Use W7.
9. Note that $(\pi-\alpha)/2+(\pi-\beta)/2+(\pi-\gamma)/2=\pi$. Use W3.
10. Use W1 for the second inequality. I found the first inequality difficult to prove.
11. Use W5 and the fact that $\tan\theta$ and $\cot\theta$ are complementary functions, i.e. $\cot(\frac{\pi}{2}-\theta)=\tan\theta$.
12. Use W4 and the same ideas as the previous problem.
13. Use Jensen's Inequality.
14. Use Jensen's Inequality.

6 Some Harder Problems

1. Use the first part of formula **7** and its related forms along with Euler's Inequality to show $0<\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=\frac{(s-a)(s-b)(s-c)}{abc}=\frac{r}{4R}\leq\frac{1}{8}$ with equality if and only if the triangle is equilateral.
2. Show that $\cos\alpha+\cos\beta+\cos\gamma=1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$.
3. Use the two previous problems to construct a proof of **W3**.
4. (**1997 Asian Pacific Mathematical Olympiad**) Let triangle ABC be inscribed in a circle and let
$$l_a=\frac{m_a}{M_a},\quad l_b=\frac{m_b}{M_b},\quad l_c=\frac{m_c}{M_c},$$
where $m_a,\ m_b,\ m_c$ are the lengths of the angle bisectors (internal to the triangle) and $M_a,\ M_b,\ M_c$ are the lengths of the angle bisectors extended until they meet the circle. Prove that
$$\frac{l_a}{\sin^2\alpha}+\frac{l_b}{\sin^2\beta}+\frac{l_c}{\sin^2\gamma}\geq 3,$$
and that equality holds if and only if ABC is an equilateral triangle.

5. **(1995 Canadian Mathematical Olympiad)** Let a, b, and c be positive real numbers. Prove that
$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

6. See the 33 problems from Bjorn Poonen's paper on inequalities.

7 References

1. E. Beckenback, R. Bellman. *An Introduction to Inequalities.* Mathematical Association of America, 1961

2. O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović, P.M. Vasić. *Geometric Inequalities.* Wolters-Noordhoff Publishing, Groningen, 1969.

3. H.S.M. Coxeter, S.L. Greitzer. *Geometry Revisited.* Mathematical Association of America, 1967.

4. Arthur Engel. *Problem Solving Strategies.* Springer, 1997.

5. M.J. Ericson, J. Flowers. *Principles of Mathematical Problem Solving.* Prentice Hall, 1999.

6. G.H. Hardy, J.E. Littlewood, G. Pólya. *Inequalities*, Second Edition. Cambridge at the University Press, 1952.

7. Richard Hoshino. *The Other Side of Inequalities in Five Parts. Mathematical Mayhem*, Volume 7, Issues 1-5, 1994.

8. H. Jacobs. *Geometry.* W.H.Freeman and Company, 1987.

9. N. Kazarinoff. *Geometric Inequalities.* Mathematical Association of America, 1961.

10. Tristan Needham. *A Visual Explanation of Jensen's Inequality. The American Mathematical Monthly*, MAA, October, 1993.

11. Bjorn Poonen. *Inequalities.* Berkeley Math Circle, November 7, 1999.

12. Paul Zeitz. *The Art and Craft of Problem Solving.* John Wiley & Sons, Inc., 1999.

If you have comments, questions or find glaring errors, please contact me by e-mail at the following address: trike@ousd.k12.ca.us