INEQUALITIES

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1. The AM-GM inequality

The most basic arithmetic mean-geometric mean (AM-GM) inequality states simply that if x and y are nonnegative real numbers, then $(x+y)/2 \geq \sqrt{xy}$, with equality if and only if $x = y$. The last phrase "with equality ... " means two things: first, if $x = y \geq 0$, then $(x+y)/2 = \sqrt{xy}$ (obvious); and second, if $(x+y)/2 = \sqrt{xy}$ for some $x, y \geq 0$, then $x = y$. It follows that if $x, y \geq 0$ and $x \neq y$, then inequality is strict: $(x+y)/2 > \sqrt{xy}$.

Here's a one-line proof of the AM-GM inequality for two variables:

$$\frac{x+y}{2} - \sqrt{xy} = \frac{1}{2}\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0.$$

The AM-GM inequality generalizes to n nonnegative numbers.

AM-GM inequality:
If $x_1, \ldots, x_n \geq 0$, then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

2. The power mean inequality

Fix $x_1, \ldots, x_n \geq 0$. For $r \neq 0$ (assume $r > 0$ if some x_i are zero), the *r-th power mean* P_r of $x_1, \ldots, x_n$ is defined to be the r-th root of the average of the r-th powers of $x_1, \ldots, x_n$:

$$P_r := \left(\frac{x_1^r + \cdots + x_n^r}{n}\right)^{1/r}.$$

This formula yields nonsense if $r = 0$, but there is a natural way to define P_0 too: it is simply defined to be the geometric mean[1]:

$$P_0 := \sqrt[n]{x_1 x_2 \ldots x_n}.$$

One also defines

$$P_\infty = \max\{x_1, \ldots, x_n\}$$

since when r is very large, P_r is a good approximation to the largest of $x_1, \ldots, x_n$. For a similar reason one uses the notation

$$P_{-\infty} = \min\{x_1, \ldots, x_n\}.$$

Here are some examples:

$$P_1 = \frac{x_1 + \cdots + x_n}{n}$$

is the arithmetic mean,

$$P_2 = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}$$

is sometimes called the *root mean square*. For $x_1, \ldots, x_n > 0$,

$$P_{-1} = \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$$

is called the *harmonic mean*.

Power mean inequality:
Let $x_1, \ldots, x_n \geq 0$. Suppose $r > s$ (and $s \geq 0$ if any of the x_i are zero). Then $P_r \geq P_s$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.

The power mean inequality holds even if $r = \infty$ or $s = -\infty$, provided that we use the

3. Convex functions

A function $f(x)$ is *convex* if for any real numbers $a < b$, each point (c, d) on the line segment joining $(a, f(a))$ and $(b, f(b))$ lies above or at the point $(c, f(c))$ on the graph of f with the same x-coordinate.

Algebraically, this condition says that

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b). \tag{1}$$

whenever $a < b$ and for all $t \in [0, 1]$. (The left hand side represents the height of the graph of the function above the x-value $x = (1-t)a + tb$ which is a fraction t of the way from a to b, and the right hand side represents the height of the line segment above the same x-value.)

Those who know what a convex set in geometry is can interpret the condition as saying that the set $S = \{(x, y) : y \geq f(x)\}$ of points above the graph of f is a convex set. Loosely speaking, this will hold if the graph of f curves in the shape of a smile instead of a frown. For example, the function $f(x) = x^2$ is convex, as is $f(x) = x^n$ for any positive *even* integer.

One can also speak of a function $f(x)$ being convex *on an interval* I. This means that the condition (1) above holds at least when $a, b \in I$ (and $a < b$ and $t \in [0, 1]$). For example, one can show that $f(x) = x^3$ is convex on $[0, \infty)$, and that $f(x) = \sin x$ is convex on $(-\pi, 0)$.

Finally one says that a function $f(x)$ on an interval I is *strictly convex*, if

$$f((1-t)a + tb) < (1-t)f(a) + tf(b)$$

whenever $a, b \in I$ and $a < b$ and $t \in (0, 1)$. In other words, the line segment connecting two points on the graph of f should lie entirely above the graph of f, except where it touches at its endpoints.

For convenience, here is a brief list of some convex functions. In these, k represents a positive integer, r, s represent real constants, and x is the variable. In fact, all of these are strictly convex on the interval given, except for x^r and $-x^r$ when r is 0 or 1.

$$\begin{aligned}
x^{2k}&, \text{ on all of } \mathbb{R} \\
x^r&, \text{ on } [0, \infty), \text{ if } r \geq 1 \\
-x^r&, \text{ on } [0, \infty), \text{ if } r \in [0, 1] \\
x^r&, \text{ on } (0, \infty), \text{ if } r \leq 0 \\
-\log x&, \text{ on } (0, \infty) \\
-\sin x&, \text{ on } [0, \pi] \\
-\cos x&, \text{ on } [-\pi/2, \pi/2] \\
\tan x&, \text{ on } [0, \pi/2) \\
e^x&, \text{ on all of } \mathbb{R} \\
r/(s + x)& \text{ on } (-s, \infty), \text{ if } r > 0
\end{aligned}$$

A sum of convex functions is convex. Adding a constant or linear function to a function does not affect convexity.

Remarks (for those who know about continuity and derivatives):
If one wants to prove rigorously that a function is convex, instead of just guessing it from the graph, it is often easier to use one of the criteria below instead of the definition of convexity.

1. Let $f(x)$ be a continuous function on an interval I. Then $f(x)$ is convex if and only if $(f(a)+f(b))/2 \geq f((a+b)/2)$ holds for all $a, b \in I$. Also, $f(x)$ is strictly convex if and only if $(f(a)+f(b))/2 > f((a+b)/2)$ whenever $a, b \in I$ and $a < b$.
2. Let $f(x)$ be a differentiable function on an interval I. Then $f(x)$ is convex if and only if $f'(x)$ is increasing on I. Also, $f(x)$ is strictly convex if and only if $f'(x)$ is strictly increasing on the interior of I.
3. Let $f(x)$ be a twice differentiable function on an interval I. Then $f(x)$ is convex if and only if $f''(x) \geq 0$ for all $x \in I$. Also, $f(x)$ is strictly convex if and only if $f''(x) > 0$ for all x in the interior of I.

4. Inequalities for convex functions

A convex function $f(x)$ on an interval $[a, b]$ is maximized at $x = a$ or $x = b$ (or maybe both).

Example (USAMO 1980/5):
Prove that for $a, b, c \in [0, 1]$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \geq 1.$$

Solution:
Let $F(a, b, c)$ denote the left hand side. If we fix b and c in $[0, 1]$, the resulting function of a is convex on $[0, 1]$, because it is a sum of functions of the type $f(a) = r/(s+a)$ and linear functions. Therefore it is maximized when $a = 0$ or $a = 1$; i.e., we can increase $F(a, b, c)$ by replacing a by 0 or 1. Similarly one can increase $F(a, b, c)$ by replacing each of b and c by 0 or 1. Hence the maximum value of $F(a, b, c)$ will occur at one of the eight vertices of the cube $0 \leq a, b, c \leq 1$. But $F(a, b, c) = 1$ at these eight points, so $F(a, b, c) \leq 1$ whenever $0 \leq a, b, c \leq 1$.

Jensen's Inequality:
Let f be a convex function on an interval I. If $x_1, \ldots, x_n \in I$, then

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 x_2 \ldots x_n}{n}\right).$$

If moreover f is strictly convex, then equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Hardy-Littlewood-Polyà majorization inequality:
Let f be a convex function on an interval I, and suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in I$. Suppose that the sequence $a_1, \ldots, a_n$ *majorizes* $b_1, \ldots, b_n$: this means that the following hold:

$$\begin{aligned}
&a_1 \geq \cdots \geq a_n \\
&b_1 \geq \cdots \geq b_n \\
&a_1 \geq b_1 \\
&a_1 + a_2 \geq b_1 + b_2 \\
&\vdots \\
&a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1} \\
&a_1 + a_2 + \cdots + a_{n-1} + a_n = b_1 + b_2 + \cdots + b_{n-1} + b_n.
\end{aligned}$$

(Note the equality in the final equation.) Then

$$f(a_1) + \cdots + f(a_n) \geq f(b_1) + \cdots + f(b_n).$$

If f is strictly convex on I, then equality holds if and only if $a_i = b_i$ for all i.

5. Inequalities with weights

Many of the inequalities we have looked at so far have versions in which the terms in a mean can be weighted unequally.

Weighted AM-GM inequality:
If $x_1, \ldots, x_n > 0$ and $w_1, \ldots, w_n \geq 0$ and $w_1 + \cdots + w_n = 1$, then

$$w_1 x_1 + w_2 x_2 + \cdots + w_n x_n \geq x_1^{w_1} x_2^{w_2} \ldots x_n^{w_n},$$

with equality if and only if all the x_i with $w_i \neq 0$ are equal.

One recovers the usual AM-GM inequality by taking equal weights $w_1 = w_2 = \cdots = w_n = 1/n$.

Weighted power mean inequality:
Fix $x_1, \ldots, x_n > 0$ and weights $w_1, \ldots, w_n \geq 0$ with $w_1 + \cdots + w_n = 1$. For any nonzero real number r, define the r-th (weighted) power mean by the formula

$$P_r := \left(\frac{w_1 x_1^r + \cdots + w_n x_n^r}{n} \right)^{1/r}.$$

Also let P_0 be the weighted geometric mean (using the same weights):

$$P_0 := x_1^{w_1} \ldots x_n^{w_n}.$$

Then P_r is an *increasing* function of $r \in \mathbb{R}$. Moreover, if the x_i with $w_i \neq 0$ are not all equal, then P_r is a *strictly increasing* function of r.

Weighted Jensen's Inequality:
Let f be a convex function on an interval I. If $x_1, \ldots, x_n \in I$, $w_1, \ldots, w_n \geq 0$ and $w_1 + \cdots + w_n = 1$, then

$$w_1 f(x_1) + \cdots + w_n f(x_n) \geq f\left(w_1 x_1 + \cdots + w_n x_n\right).$$

If moreover f is strictly convex, then equality holds if and only if all the x_i with $w_i \neq 0$ are equal.

6. Symmetric function inequalities

Given numbers $a_1, \ldots, a_n$ and $0 \leq i \leq n$, the i-th *elementary symmetric function* σ_i is defined to be the coefficient of x^{n-i} in $(x + a_1) \ldots (x + a_n)$. For example, for $n = 3$,

$$\begin{aligned}
\sigma_0 &= 1 \\
\sigma_1 &= a_1 + a_2 + a_3 \\
\sigma_2 &= a_1 a_2 + a_2 a_3 + a_3 a_1 \\
\sigma_3 &= a_1 a_2 a_3.
\end{aligned}$$

The i-th *elementary symmetric mean* S_i is the arithmetic mean of the monomials appearing in the expansion of σ_i; in other words, $S_i := \sigma_i / \binom{n}{i}$. In the example above,

$$\begin{aligned} S_0 &= 1 \\ S_1 &= \frac{a_1 + a_2 + a_3}{3} \\ S_2 &= \frac{a_1 a_2 + a_2 a_3 + a_3 a_1}{3} \\ S_3 &= a_1 a_2 a_3. \end{aligned}$$

Newton's inequality:
For any real numbers $a_1, \ldots, a_n$, we have $S_{i-1} S_{i+1} \leq S_i^2$.

Maclaurin's inequality:
For $a_1, \ldots, a_n \geq 0$, we have

$$S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \cdots \geq \sqrt[n]{S_n}.$$

Moreover, if the a_i are positive and not all equal, then the inequalities are all strict.

7. More inequalities

Cauchy(-Schwartz-Buniakowski) inequality:
If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are real numbers, then

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \geq (x_1 y_1 + \cdots + x_n y_n)^2.$$

Chebychev's inequality:
If $x_1 \geq \cdots \geq x_n \geq 0$ and $y_1 \geq \cdots \geq y_n \geq 0$, then

$$\frac{x_1 y_1 + \cdots + x_n y_n}{n} \geq \left(\frac{x_1 + \cdots + x_n}{n} \right) \left(\frac{y_1 + \cdots + y_n}{n} \right)$$

with equality if and only if one of the sequences is constant.

Chebychev's inequality with three sequences:
If $x_1 \geq \cdots \geq x_n \geq 0$, $y_1 \geq \cdots \geq y_n \geq 0$, and $z_1 \geq \cdots \geq z_n \geq 0$, then

$$\frac{x_1 y_1 z_1 + \cdots + x_n y_n z_n}{n} \geq \left(\frac{x_1 + \cdots + x_n}{n} \right) \left(\frac{y_1 + \cdots + y_n}{n} \right) \left(\frac{z_1 + \cdots + z_n}{n} \right)$$

with equality if and only if at least two of the three sequences are constant or one of the sequences is all zero.

You can probably guess what the four-sequence Chebychev inequality looks like.

Hölder's inequality:
Let $a_1, \ldots, a_n, b_1, \ldots, b_n, \alpha, \beta > 0$ and suppose that $\alpha + \beta = 1$. Then

$$(a_1 + \cdots + a_n)^\alpha (b_1 + \cdots + b_n)^\beta \geq (a_1^\alpha b_1^\beta + \cdots + a_n^\alpha b_n^\beta),$$

with equality if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

Jensen's extension of Hölder's Inequality:
Suppose $a_1, \dots, a_n, b_1, \dots, b_n, \dots, \ell_1, \dots, \ell_n, \alpha, \beta, \dots, \lambda > 0$, and $\alpha + \beta + \cdots + \lambda \geq 1$. Then

$$\left(\sum_{i=1}^{n} a_i \right)^{\alpha} \left(\sum_{i=1}^{n} b_i \right)^{\beta} \cdots \left(\sum_{i=1}^{n} \ell_i \right)^{\lambda} \geq \sum_{i=1}^{n} \left(a_i^{\alpha} b_i^{\beta} \dots \ell_i^{\lambda} \right).$$

Rearrangement inequality:
Suppose $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$ are real numbers. If π is a permutation of $1, 2, \dots, n$, then

$$a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1 \leq a_1 b_{\pi(1)} + \cdots + a_n b_{\pi(n)} \leq a_1 b_1 + \cdots + a_n b_n.$$

Minkowski's inequality:
Suppose $a_1, \dots, a_n, b_1, \dots, b_n \geq 0$, and r is a real number. If $r > 1$, then

$$\sqrt[r]{a_1^r + \cdots + a_n^r} + \sqrt[r]{b_1^r + \cdots + b_n^r} \geq \sqrt[r]{(a_1 + b_1)^r + \cdots + (a_n + b_n)^r}.$$

If $0 < r < 1$, then the inequality is reversed.

Bernoulli's inequality:
If $x > -1$ and $0 < a < 1$, then

$$(1 + x)^a \leq 1 + ax,$$

with equality if and only if $x = 0$. The inequality reverses for $a < 0$ or $a > 1$.

8. Problems

There are a lot of problems here. Just do the ones that interest you.

1. Prove that for any $a, b, c > 0$,
$$(a + b)(b + c)(c + a) \geq 8abc,$$
and determine when equality holds.
2. Prove $n! < \left(\frac{n+1}{2}\right)^n$ for all integers $n > 1$.
3. Prove that the sum of the legs of a right triangle never exceeds $\sqrt{2}$ times the hypotenuse of the triangle.
4. Prove $2\sqrt{x} \geq 3 - 1/x$ for $x > 0$.
5. Prove that if $a > b > 0$ and $n \geq 1$ is an integer, then
$$a^n - b^n > n(a - b)(ab)^{(n-1)/2}.$$
6. Let E be the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$
for some $a, b, c > 0$. Find, in terms of a, b, and c, the volume of the largest rectangular box that can fit inside E, with faces parallel to the coordinate planes.
7. Among all planes passing through a fixed point (a, b, c) with $a, b, c > 0$ and meeting the positive parts of the three coordinate axes, find the one such that the tetrahedron bounded by it and the coordinate planes has minimal area.
8. Among all rectangular boxes with volume V, find the one with smallest surface area.
9. Now consider "open boxes," with only five faces. Again find the one with smallest surface area with a given volume V.

10. Let T be the tetrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$ for some $a,b,c>0$. Let V be the volume of T, and let ℓ be the sum of the lengths of the six edges of T. Prove that
$$V \leq \frac{\ell^3}{6(3+3\sqrt{2})^3}.$$
11. Let $g = \sqrt[n]{a_1 \dots a_n}$ be the geometric mean of the numbers $a_1, \dots, a_n > 0$. Prove that
$$(1+a_1)(1+a_2)\dots(1+a_n) \geq (1+g)^n.$$
12. Suppose $x,y,z>0$ and $x+y+z=1$. Prove that
$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64.$$
13. Show that one can derive the AM-GM inequality for positive numbers from Jensen's inequality with $f(x) = -\log x$.
14. Prove $x^x \geq \left(\frac{x+1}{2}\right)^{x+1}$ for $x>0$. (Hint: the function $x \log x$ is convex on $(0,\infty)$.)
15. Use Jensen's inequality to show that among all convex n-gons inscribed in a fixed circle, the regular n-gons have the largest perimeter.
16. Given $a,b,c,p,q,r>0$ with $p+q+r=1$, prove
$$a+b+c \geq a^p b^q c^r + a^r b^p c^q + a^q b^r c^p.$$
17. Show that by taking some of the a_i to be equal in the AM-GM inequality, one can deduce the weighted AM-GM inequality at least in the case where the weights are nonnegative rational numbers. (To deduce from this the general weighted AM-GM inequality, one can then use a limit argument.) Can one similarly deduce the weighted power mean inequality and weighted Jensen's inequality from the unweighted versions?
18. Prove that if a,b,c are sides of a triangle, then
$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \leq a^a b^b c^c.$$
19. Given $a,b,c,d>0$ such that $(a^2+b^2)^3 = c^2+d^2$, prove
$$\frac{a^3}{c} + \frac{b^3}{d} \geq 1.$$
20. What well-known inequality does one obtain by taking only the end terms in Maclaurin's inequality?
21. Prove
$$(bc+ca+ab)(a+b+c)^4 \leq 27(a^3+b^3+c^3)^2$$
for $a,b,c \geq 0$.
22. Prove that if $x,y,z,a,b,c>0$, then
$$\frac{x^4}{a^3} + \frac{y^4}{b^3} + \frac{z^4}{c^3} \geq \frac{(x+y+z)^4}{(a+b+c)^3}.$$
23. Prove that if a,b,c are sides of a triangle, then
$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$
24. Derive Chebychev's inequality from the rearrangement inequality.
25. Derive the 3-sequence Chebychev inequality from the 2-sequence Chebychev inequality.

26. Suppose that $0 \le \theta_1, \dots, \theta_n \le \pi/2$ and $\theta_1 + \cdots + \theta_n = 2\pi$. Prove that
$$4 \le \sin(\theta_1) + \cdots + \sin(\theta_n) \le n \sin(2\pi/n).$$
27. Show that Jensen's inequality is a special case of the Hardy-Littlewood-Polyà majorization inequality.
28. What is the geometric meaning of Minkowski's inequality when $r = 2$ and $n = 3$?
29. (IMO 1975/1) Let x_i, y_i $(i = 1, 2, \dots, n)$ be real numbers such that
$$x_1 \ge x_2 \ge \cdots \ge x_n \qquad \text{and} \qquad y_1 \ge y_2 \ge \cdots \ge y_n.$$
Prove that if $z_1, z_2, \dots, z_n$ is any permutation of $y_1, y_2, \dots, y_n$, then
$$\sum_{i=1}^{n} (x_i - y_i)^2 \le \sum_{i=1}^{n} (x_i - z_i)^2.$$
30. (USAMO 1977/5) Suppose $0 < p < q$, and $a, b, c, d, e \in [p, q]$. Prove that
$$(a + b + c + d + e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \le 25 + 6\left(\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}\right)^2$$
and determine when there is equality.
31. (IMO 1964/2) Prove that if a, b, c are sides of a triangle, then
$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \le 3abc.$$
32. (USAMO 1981/5) If x is a positive real number, and n is a positive integer, prove that
$$\lfloor nx \rfloor \ge \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \cdots + \frac{\lfloor nx \rfloor}{n},$$
where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.
33. Make your own inequality problems and give them to your friends (or enemies, depending on the difficulty!)

Many of the problems above were drawn from notes from the U.S. training session for the International Mathematics Olympiad. Others are from the USSR Olympiad Problem Book. Many of the inequalities themselves are treated in the book "Inequalities" by Hardy, Littlewood, and Polyà, which is a good book for further reading.

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