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COMPLEX NUMBERS IN GEOMETRY

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1. Let O be a point in the plane of $\triangle ABC$. Points A_1, B_1, C_1 are the images of A, B, C under symmetry with respect to O. Prove that the circumcircles of $\triangle ABC$, $\triangle A_1B_1C$, $\triangle A_1BC_1$ and $\triangle AB_1C_1$ pass through the same point.
2. $ABCD$ is inscribed in circle k with center O. The perpendiculars through A to the sides AB and AD intersect sides CD and BC in points M and N, respectively. Prove that line MN passes through O.
3. Let AB and CD be two chords in circle k. Let M be the intersection of the perpendiculars from A to AB and from C to CD; and let N be the intersection of the perpendiculars from B to AB and from D to CD. Prove that line MN passes through the intersection of lines BC and AD (if these two intersect), or is parallel to them (if they are parallel to each other).
4. (Simpson) Prove that the feet of the perpendiculars dropped from a point on the circumcircle k of $\triangle ABC$ to the sides of the triangle are collinear (cf. Fig. 4a.)

FIGURE 4A

FIGURE 4B

5. (Simpson) More generally, let S be the area of $\triangle ABC$, R – the circumradius, and d – the radius of a circle ϵ concentric to k. Let A_1, B_1 and C_1 be the feet of the perpendiculars dropped from an arbitrary point on ϵ to the sides of $\triangle ABC$. Prove that the area S_1 of $\triangle A_1B_1C_1$ is given by the formula $S_1 = \frac{1}{4}S\left|1 - \frac{d^2}{R^2}\right|$. In particular, when $\epsilon = k$, then $S_1 = 0$, and hence A_1, B_1 and C_1 are collinear (cf. Fig. 4b.)

THESE NOTES ARE FROM THE MATHEMATICAL OLYMPIAD SUMMER PROGRAM'99.

6. (Newton) Quadrilateral $PQRS$ is circumscribed around circle k with center O. Prove that the midpoints of the diagonals of $PQRS$ and point O are collinear.
7. (Gauss) If the two pairs of opposite sides of a quadrilateral intersect, prove that the midpoint of the segment connecting their intersection points lies on the line through the midpoints of the diagonals.
8. Let H be the orthocenter of $\triangle A_1A_2A_3$. The circle with diameter A_3H intersects sides A_2A_3 and A_1A_3 in points P and Q respectively. Prove that the tangents at P and Q to k intersect each other at the midpoint of side A_1A_2.
9. Let k with center O be the circumcircle of $\triangle A_1A_2A_3$. Let A_3O intersect side A_1A_2 in point M; let A_3P_3 be the altitude, and E be the Euler center of 9 points for $\triangle A_1A_2A_3$. Prove that N, E and P_3 are collinear.
10. In acuteangled $\triangle ABC$, the orthocenter H divides the altitude BD in ratio $BH : HD = 3 : 1$. Prove that $\angle AKC = 90°$, where K is the midpoint of BD.
11. The angles of $\triangle ABC$ form a geometric series with ratio 2. Prove that the midpoints of its sides and the feet of its altitudes are vertices of a regular 7–gon.
12. In the plane of $\triangle ABC$ there exist two points U and V such that $\triangle AUV \sim \triangle VBU \sim \triangle UVC$, and these three triangles are equally oriented. Prove that these three triangles are also similar to $\triangle ABC$.
13. B, C and P lie on a circle k with center O. The tangents to k at B and C intersect in A; the perpendicular to AP at P intersects OB and OC in D and E. DM and EN are the perpendiculars from D and E to OA. Prove that
    (a) $\triangle OAD \sim \triangle OEA$.
    (b) M and N are images of each other under inversion with respect to k.
14. Given $\triangle A_1A_2A_3$ with different lengths of the sides, let M_i be the midpoint of the side opposite to A_i, T_i – the point of tangency of this side with the incircle k, S_i – the symmetric point of T_i with respect to the angle bisector of $\angle A_i$ $(i = 1, 2, 3.)$ Prove that lines M_1S_1, M_2S_2 and M_3S_3 intersect in a point on k.
15. In $\triangle ABC$ prove that the angle bisector of $\angle A$, the midsegment parallel to AC, and the line joining the tangent points of the incircle with sides BC and CA, are concurrent.
16. In rectangle $ABCD$, the angle bisector of $\angle B$ intersects diagonal AC and side AD in E and F, respectively. A line through E parallel to AB intersects diagonal BD in K. Prove that line FK is perpendicular to AC.
17. Given a non-rectangular parallelogram $ABCD$, a circle k with diameter AC intersects lines AB and AD in points M and N (other than A.) Prove that lines BD, MN and the tangent at C to k are concurrent (or all parallel).

18. Let P and C lie on a semicircle s with diameter AB, so that arcs BC and CD are equal. If $AC \cap BP = E$ and $AD \cap CP = F$, then prove $EF \perp AD$.

19. $ABCD$ is circumscribed around circle k. Let l_1 and l_2 be two arbitrary tangents to k, different from the sides of $ABCD$. The distances from A, B, C, D to l_i are a_i, b_i, c_i, d_i $(i = 1, 2.)$ Prove that $a_1b_2c_1d_2 = a_2b_1c_2d_1$.

20. Let t_A and t_B be the tangent to circle k at two diametrically opposite points A and B on k. Through point C on t_A $(C \neq A)$ draw two chords D_1E_1 and D_2E_2 in k. Prove that the rays $\overrightarrow{AD_1}$ and $\overrightarrow{AD_2}$ cut a segment from t_B of length equal to that of the segment cut on t_B by the rays $\overrightarrow{AE_1}$ and $\overrightarrow{AE_2}$.

21. On a semicircle s with diameter AB take arbitrary points C and D. Points P, Q and R are the midpoints of AC, CD and BD. Through points P and R draw lines perpendicular to AQ and BQ, respectively, and let them intersect the tangents to s at A and B at points S and T, respectively. Prove that ST and CD are parallel.

22. Let O and H be the circumcenter and orthocenter of $\triangle A_1A_2A_3$. Lines A_1H, A_2H and A_3H intersect the circumcircle k of $\triangle A_1A_2A_3$ in points Q_1, Q_2 and Q_3, respectively. Prove that the lines through Q_1, Q_2 and Q_3, parallel correspondingly to OA_1, OA_2 and OA_3, are concurrent.

23. Prove that on the circumcircle k of $\triangle ABC$ there exist exactly three points X $(\neq A)$ with the following property: X is the midpoints of the segment cut by the arms of $\angle BAC$ on the tangent through X to k. Prove also that the orthocenter of the triangle with vertices these three points is the midpoint of side BC (cf. Fig. 23.)

FIGURE 23 FIGURE 24 FIGURE 27

24. (Moscow'97 X) Each side of a polygon is extended to twice its length at one of its ends while going around the polygon in counterclockwise direction. If the newly obtained ends of segments form a regular polygon, prove that the original polygon is also regular (cf. Fig. 24.)

25. (Sorovska'97 IX) Point C lies inside $\triangle ABD$ so that $\triangle ABC$ is right and isosceles with hypothenuse $AB = 2$, and $CD = 1$. On the ray through C and perpendicular to and intersecting AD draw segment $CK = AD$. Similarly, on the ray through C and perpendicular to and intersecting BD draw segment $CM = BD$. Prove that points K, D and M are collinear.
26. (MOSP'99 test) Let H, O and R be the orthocenter, circumcenter and circumradius of $\triangle ABC$. Let A_1, B_1 and C_1 be the reflections of A, B and C across lines BC, CA and AB. Prove that A_1, B_1 and C_1 are collinear iff $OH = 2R$.
27. (IMO'99) Circles k_1 and k_2 lie inside circle k and are tangent to k at respective points M and N. k_1 passes through the center of k_2. The common chord of k_1 and k_2 hits k at A and B; lines MA and MB intersect k_1 again at C and D. Prove that CD is tangent to k_2. (cf. Fig. 27.)

Useful Formulas

28. $\triangle ABC$ and $\triangle A_1B_1C_1$ are similar and equally oriented (cf. Fig. 28) iff

$$\frac{(b-a)}{(c-a)} = \frac{(b_1-a_1)}{(c_1-a_1)}.$$

29. Three distinct points A, B and C are collinear iff $\dfrac{(b-a)}{(c-a)} = \dfrac{(\bar{b}-\bar{a})}{(\bar{c}-\bar{a})}$.

Figure 28

Figure 30

Definition. Let $a = x + iy$ and $b = u + iv$ with $x, y, u, v \in \mathbb{R}$. Define

$$\det\{ab\} = xv - yu = \det \begin{vmatrix} x & y \\ u & v \end{vmatrix}$$

Note that $\det\{ab\}$ equals half of the oriented area of $\triangle OAB$, where the sign is $+$ if $\angle AOB \leq 180°$ in counterclockwise direction, and $-$ otherwise. *Determinants*, as defined above, are anticommutative, distributive, and linear with respect to multiplying by real numbers.

30. Three points A, B and C are collinear (cf. Fig. 30) iff

$$\det\{ab\} + \det\{bc\} + \det\{ca\} = 0.$$

31. Lines AB and CD are perpendicular iff

$$\frac{(b-a)}{(c-d)} = -\frac{(\overline{b}-\overline{a})}{(\overline{c}-\overline{d})}.$$

32. If A and B are on the unit circle, then a point M lies on line AB iff it satisfies the equation (cf. Fig. 32):

$$a + b = m + \overline{m}ab.$$

FIGURE 32-33 FIGURE 34 FIGURE 35 FIGURE 36

33. If T lies on the unit circle k, then the tangent to k at Z is described by the equation (cf. Fig. 33):

$$2t = z + \overline{z}t^2.$$

34. If four points A, B, C and D lie on the unit circle, then the intersection point Z of lines AB and CD is given by (cf. Fig. 34):

$$z = \frac{ab(c+d) - cd(a+b)}{ab - cd} = \frac{(\overline{c}+\overline{d}) - (\overline{a}+\overline{b})}{\overline{c}\overline{d} - \overline{a}\overline{b}}.$$

35. Let A, B and T lie on the unit circle k. The intersection point Z of line AB and the tangent to k at T is given by (cf. Fig. 35):

$$z = \frac{t(2ab - ta - tb)}{ab - t^2}.$$

36. If points A and B lie on the unit circle k and are not diametrically opposite, then the intersection point Z of the tangents to k at A and B, and its inverse image Z' with respect to k are given by (cf. Fig. 366):

$$z = \frac{2ab}{a+b} = \frac{2}{\overline{a}+\overline{b}}, \quad z' = \frac{a+b}{2}.$$

37. Let A and B lie on the unit circle and D is an arbitrary point in the plane. Then the foot Z of the perpendicular from D to line AB is given by (cf. Fig. 37):

$$z = \frac{1}{2}(a + b + d - ab\overline{d})\cdot$$

38. Let T_1, T_2, S_1 and S_2 lie on the unit circle k. Then chords T_1S_1 and T_2S_2 are parallel iff $t_1s_1 = t_2s_2$ (cf. Fig. 38.)

FIGURE 37 FIGURE 38 FIGURE 39 FIGURE 40

39. Four points A, B, C and D in the plane are concyclic iff (cf. Fig. 39):

$$\frac{(a-b)(c-d)}{(a-d)(c-b)} \in \mathbb{R}.$$

40. If $\triangle ABC$ is inscribed in the unit circle k, then its orthocenter H and its center E of the Euler circle of 9 points are given by (cf. Fig. 40):

$$h = a + b + c, \quad e = (a + b + c)/2.$$

41. The perpendicular bisector of segment AB has equation (cf. Fig. 41):

$$z(\overline{a} - \overline{b}) + \overline{z}(a - b) = a\overline{a} - b\overline{b}.$$

If A and B lie on the unit circle, then the above is $z = \overline{z}ab$.

FIGURE 41 FIGURE 42 FIGURE 43

42. If vertex C of $\triangle ABC$ is the center of the complex coordinate system, then the circumcenter Z of $\triangle ABC$ is given by (cf. Fig. 42):

$$z = \frac{ab(\overline{a} - \overline{b})}{b\overline{a} - a\overline{b}}\cdot$$

43. Let AB be a chord on the unit circle, and let C be an arbitrary point. Then the reflection C_1 of C across line AB is given by (cf. Fig. 43):

$$c_1 = a + b - \overline{c}ab.$$

Short Appendix on Complex Numbers

The points in the usual coordinate plane P can be thought of as complex numbers: the point $A = (a, b)$ can be thought of as the complex number $z = a + bi$ with $a, b \in \mathbb{R}$. Thus, the x-coordinate of A corresponds to the *real part* of z: $\mathbb{R}\mathrm{e}(z) = a$, and the y-coordinate of A corresponds to the *imaginary part* of z: $\mathbb{I}\mathrm{m}(z) = b$. Recall how we add and subtract complex numbers: this corresponds exactly to addition and subtraction of vectors originating at (0,0) in the plane. For instance, if $z_1 = a_1 + b_1 i$, then $z + z_1 = (a + a_1) + (b + b_1)i$; this corresponds exactly to what would happen if we add two vectors $\vec{v}$ and $\vec{v}_1$ which start at the origin and end in (a, b) and (a_1, b_1), respectively: $\vec{v} + \vec{v}_1$ would start at the origin and end in $(a + a_1, b + b_1)$.

Multiplication of complex numbers can be also translated in terms of vectors in the plane. To multiply z and z_1 from above, we perform the usual algebraic manipulations:

$$z \cdot z_1 = (a + bi) \cdot (a_1 + b_1 i) = aa_1 + ab_1 i + ba_1 i + bb_1(i^2) = (aa_1 - bb_1) + (ab_1 + ba_1)i.$$

The resulting "vector" $\vec{v}'$ from this multiplication corresponds to $(aa_1 - bb_1, ab_1 + ba_1)$, and it can be interpreted geometrically from the starting vectors $\vec{v}$ and $\vec{v}_1$. I urge you to check in a few simple examples that $\vec{v}'$ can be described as follows: add the angles that $\vec{v}$ and $\vec{v}_1$ form with the x-axis – this is going to be direction of $\vec{v}'$; for the length of $\vec{v}'$, take the product of the lengths of $\vec{v}$ and $\vec{v}_1$. (Hint: use the so-called "polar form" of vectors and some simple trigonometric identities.)

We introduce here one further notion: the *conjugate* of a complex number. If $z = a + bi$ is a complex number, then the conjugate of z, denoted by $\overline{z}$, is simply the complex number obtained from be z by switching the sign of z's imaginary part: $\overline{z} = a - bi$. Geometrically, the points (a, b) and $(a, -b)$ are reflections of each other across the x-axis. The "miraculous" property of conjugates is that their product is always a *real* number:

$$z \cdot \overline{z} = (a + bi) \cdot (a - bi) = a^2 + b^2 \in \mathbb{R}.$$

Note that a complex number z is real iff $z = \overline{z}$ (i.e. z lies on the x–axis), and it is purely imaginary iff $z = -\overline{z}$ (i.e. z lies on the y–axis).