Addenda to Archimedes

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1 Archimedes Cattle Problem

At the talk on Archimedes, I presented the equations for Archimedes famous Cattle Problem. I also gave the first four digits for the number, W, of white bulls and the first four digits for the total number of cattle in the herd, so you could check your answers. The numbers I gave, taken from Dijksterhuis [**1**] were both incorrect in the fourth digit. The correct digits are $W = 1596\ldots$ and a total of $7760\ldots$. For reference the equations I put on the board were, using Dijksterhuis' notation.

$$\begin{aligned}
W &= (\frac{1}{2} + \frac{1}{3})Z + B \\
Z &= (\frac{1}{4} + \frac{1}{5})P + B \\
P &= (\frac{1}{6} + \frac{1}{7})W + B \\
w &= (\frac{1}{3} + \frac{1}{4})(Z + z) \\
z &= (\frac{1}{4} + \frac{1}{5})(P + p) \\
p &= (\frac{1}{5} + \frac{1}{6})(B + b) \\
b &= (\frac{1}{6} + \frac{1}{7})(W + w)
\end{aligned}$$

In addition, it required that $W + Z$ be a perfect square, n^2, and $P + B$ be a triangular number, $\frac{m(m+1)}{2}$. For a discussion of the solution to the first seven equations see the first problem in *100 Great Problems of Elementary Mathematics* by Heinrich Dörrie [**2**]. Gabriel Carroll mentioned to me that this book was one of the first mathematics books he read. A "more complete" formulation of the problem was discovered in a Greek manuscript in 1773 in the Wolfenbüttel library. A translation from the Greek to English (via German, since Dörrie's book was originally in German), in poetic form, made up of twenty-two distichs, is also included along with some other historical comments a general discussion of the solution to the complete problem. If you go searching, you will find problem three in this book is a problem on cows and fields by Newton, but it is not nearly as interesting. However, the book is a goldmine of interesting problems in mathematics by the great mathematicians of the past, with very succinctly written solutions.

In Heath [**3**], an alternate interpretation of the bulls forming a "square" came about by considering bulls to be longer than they are wide, and so seeking an answer where the bulls are closely packed to form a “square figure”, rather than requiring the number of bulls, $W+Z$, to be a perfect square. This problem is easier and is known as *Wurm's Problem.* It is solved in Heath. Heath then goes on to discuss the solution to the complete problem which leads to the Pellian equation $t^2 - 4729494u^2 = 1$. This type of problem was discussed at the Berkeley Math Circle last year. For a nice introduction via the discovery method, see the Power Round of the Polya Contest held at Gunn High School (October 30, 1999). Unfortunately, Heath also has the incorrect fourth digit for W. This may be where Dijksterhuis got his information, since it is the same error.

In 1889, A.H. Bell, a civil engineer, and two friends formed the Hillsboro, Illinois, Mathematical Club and started the computation of the solution to the complete problem. After four years they computed the first 32 left-hand digits and the last 12 right-hand digits for each variety of bulls and cows, as well as the total number of cattle in the herd. This is detailed in Albert H. Beiler's *Recreations in the Theory of Numbers* [**4**]. However, there is a misprint in the book. The value printed for the variable t is actually the value of t^2. A truncated version of the problem in prose is also given. The discussion of the problem appears in the chapter entitled *The Pellian* where you will find out why Pell, who had almost nothing to do with solving this type of equation has his name gloriously attached to it. There is also a fairly clear presentation of the method of solution via continued fractions.

The first complete listing of the solutions to the problem was given in 1965 by the Canadian mathematicians H.C. Williams, R.A. German, and C.R. Zarnke, who computed it using a computer. The computer printout is on deposit among Unpublished Mathematical Tables at the University of Maryland. This showed that the last two of Bell's thirty-two left-hand digits were incorrect. In 1980, Harry L. Nelson of the Lawrence Livermore National Laboratory recast the problem in code suitable for exercising the newly delivered CRAY-1 computer. The computation of the solution, together with extensive checking was done in ten minutes. Since this was not of sufficient length for the purpose desired, the code went on to find five additional solutions, the largest of which has well over a million digits. All 206,545 digits of the smallest solution taking up over 46 computer pages (64 rows of 70 digits) are printed at one-third actual size, four-to-a-page, in the *Journal of Recreational Mathematics* [**5**]. See the recent article by Vardi [**6**] for a beautiful discussion of personal computer techniques and a prose translation of the problem.

2 References

1. E. J. Dijksterhuis, *Archimedes.* Princeton University Press, 1987.

2. Heinrich Dörrie. *100 Great Problems of Elementary Mathematics.* Dover Publications, Inc.,1965.

3. T.L. Heath, *The Works of Archimedes* . Dover Publications,Inc. Reissue of the 1897 edition with the 1912 Supplement on *The Method.*

4. Albert H. Beiler. *Recreations in the Theory of Numbers.* Dover Publications,Inc., 1964.

5. Harry L. Nelson. *Journal of Recreational Mathematics*, pp 162-176, Vol. 13 No. 3, 1980-81.

6. Ilan Vardi. Archimedes Cattle. *The American Mathematical Monthly*, pp 305-319 Vol. 105 No. 4, April 1998. Mathematical Association of America.