Archimedes and the Arbelos

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Berkeley Math Circle January 16, 2000

1 History and Sources

Archimedes lived from 287 BC until he was killed by a Roman soldier in 212 BC. He is usually considered to be one of the three greatest mathematicians of all time, the other two being Newton and Gauss. The extant works of Archimedes are readily available today in Heath [2] and Dijksterhuis [3]. Both contain the works with extensive notes and historical information. They both can be somewhat difficult to plow through at times. To ease the burden, a new book by Sherman Stein [3] came out last year that is accessible to a much broader audience. It fact the only prerequisite is high school algebra and geometry. A partial listing of the works that have not been lost is as follows : On the Equilibrium of Planes, Quadrature of the Parabola, On the Sphere and Cylinder, On Spirals, On Conoids and Spheroids, On Floating Bodies, Measurement of a Circle, The Sand-reckoner, and The Book of Lemmas.

2 The Arbelos ($\acute{\alpha}\varrho\beta\eta\lambda o\varsigma$)

The arbelos consists of of three points A, B, C which are collinear, together with the semicircles ADB, AXC and CYB as shown in Figure 1. It was so named because of its shape, which resembles a shoemaker's knife, or $\acute{\alpha}\varrho\beta\eta\lambda o\varsigma$. It engaged the attention of no less a mathematician than Archimedes. He played with this figure for fun, which is an excellent reason for doing mathematics. His theorems about the figure are contained in the Book of Lemmas which come to us in the form of an Arabic manuscript that details what Archimedes proved.

In the figure $\overline{CD}$ has been added to the figure tangent to the two small semicircles. $\overline{AD}$ intersects a small semicircle at X and $\overline{BD}$ intersects the other small semicircle at Y. $\overline{XY}$ intersects $\overline{CD}$ at P. For fun and relaxation, try proving the following statements.

1. The area of the arbelos is equal to the area of the circle with diameter $\overline{CD}$.
2. $\overline{XY}$ and $\overline{CD}$ bisect each other.
3. $\overline{XY}$ is tangent to the small semicircles.
4. If two circles are tangent at A, and if $\overline{BD}$, $\overline{EF}$ are parallel diameters in the circles, then A, D, and F are collinear. (This is Proposition 1 in *The Book of Lemmas*).

5. In Figure 2, the two semicircles inscribed in regions ACD and BCD have equal radii. Find this radius in terms of the radii of the three semicircles that form the arbelos. (This is Proposition 5 in *The Book of Lemmas*).

6. Construct the twin circles in a given arbelos with a straightedge and compass.

7. In Figure 3, find the radius of the circle tangent to all three of the semicircles that form the arbelos in terms of the radii of these three semicircles. (This is Proposition 6 in *The Book of Lemmas*).

8. (**East Bay Mathletes, April 1999**) The small circle in the figure is tangent to all three of the semicircles. If the diameters of the two small semicircles are 10 and 6 units, then what is the radius of this small circle? Answer as a fraction in lowest terms. (Hint: It can be easily proved by inversion that the distance of the center of the small circle from the base is equal to the diameter of the small circle.) See the next problem.

9. (**Pappus**) In Figure 4, the chain of inscribed circles, C_n, in an arbelos has the following property. The distance from the diameter of the largest semicircle to the center of the nth circle in the chain, C_n, is exactly equal to d_n, where d_n is the diameter of C_n. (Hint: Use inversion in a circle orthogonal to C_n.)

10. The centers of the chain of inscribed circles in an arbelos lie on an ellipse with foci at the centers of the two semicircles to which each circle of the chain is tangent.

For a discussion of how Pappus proved this result, (The technique of inversion was not known until the 19th Century.) see the article *How Did Pappus Do It?* by Leon Bankoff on pages 112-118 of the book *The Mathematical Gardner* edited by David Klarner and published by Pridle, Weber & Schmidt in 1981. The book was written as a tribute to Martin Gardner. Martin Gardner also wrote a column on the arbelos for *Scientific American* in January 1979. The column is also available in *Fractal Music, Hypercards and More...* which reprints his columns from 1978 and 1979.

3 Some Generalizations

1. (**The twins are triplets**) In Figure 5, let P and Q be the points where the circle inscribed in the arbelos is tangent to the two smaller semicircles, and let R be the point where the two smaller semicircles are tangent. The circle containing P, Q and R has a radius equal to that of the twin circles, namely $r_1 r_2 / r$. See [**8**].

2. (**Quadruplets**) The largest circle inscribed in the circular segment of the largest semicircle formed by the chord $\overline{EF}$ tangent to the other two semicircles also has the same radius as the twins. See Figure 6. Note that the point of tangency for this circle is the point D of the common internal tangent to the smaller semicircles of the arbelos, $\overline{CD}$. The story doesn't end here. There are infinite families of circles in and around the arbelos with the same radius as the twins. See [**10**] and [**11**].

3. In Figure 7, a chain of circles, inscribed in two internally tangent semicircles with collinear endpoints, begins with a circle tangent to the line of centers of the semicircles.

The distance of the center of the nth circle in the chain from the line of centers is $(2n-1)r_n$ where r_n is the radius of the nth circle.

4. If the radius of the largest semicircle in an arbelos is k times the radius of the smallest semicircle, where k is an integer, then at least one pair of circles in the chain will have their centers on a line parallel to the base of the arbelos. The circles paired in this manner are those whose order numbers in the chain have a product equal to $k(k-1)$. See *On a Generalization of the Arbelos* by M. G. Gaba in the *American Mathematical Monthly*, vol 47, Jan. 1940, pp 19-24.

5. (**Valentine**) Let r, r_1, r_2 be the radii of the semicircles forming an arbelos and let ρ be the radius of the inscribed circle. In Figure 8, prove the ratio of the area of the arbelos to the area of the heart equals $\frac{\rho}{r} = \frac{r^2 - r_1^2 - r_2^2}{r^2 + r_1^2 + r_2^2}$. See [**9**] for ten proofs by Charles W. Trigg. See [**4**] for seven proofs that $\rho = \frac{rr_1r_2}{r_1^2 + r_1r_2 + r_2^2}$.

4 References

1. Sherman Stein. *Archimedes: What Did He Do Besides Cry Eureka.* pp 7-25. Mathematical Association of America, 1999.

2. T.L. Heath, *The Works of Archimedes* .Dover Pubilications,Inc. Reissue of the 1897 edition with the 1912 Supplement on *The Method.*

3. E. J. Dijksterhuis, *Archimedes.* Princeton University Press, 1987.

4. Problem Solutions. pp 112-115 Nov-Dec 1952 *Mathematics Magazine*, Mathematical Association of America.

5. Brother L. Raphael,F.S.C. *The Shoemaker's Knife*, *Mathematics Teacher*, April 1973.

6. Ogilvy, C. S. *Excursions in Geometry* Oxford University Press, 1969

7. Harold Jacobs. *Geometry.* First Edition page 534. W.H.Freeman and Company, 1974.

8. Leon Bankoff. *Are the Twin Circles of Archimedes really Twins?.* pp 214-218 Vol 47 (1974) *Mathematics Magazine*, Mathematical Association of America.

9. Charles W. Trigg. *How Do I Love Thee? Let Me Count the Ways. Eureka* now *Crux Mathematicorum*, pp 217-224 Vol. 3 No. 8 October 1977.

10. Leon Bankoff. *The Marvelous Arbelos.* pp 247-253 *The Lighter Side of Mathematics*, Mathematical Association of America, 1994.

11. Clayton W. Dodge, Thomas Shoch, Peter Y. Woo, Paul Yiu *Those Ubiquitous Archimedean Circles.* pp 202-213 Vol 72 No 3 June 1999 *Mathematics Magazine*, Mathematical Association of America.

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