BAY AREA MATHEMATICAL ADVENTURE

GEOMETRIC PUZZLES AND CONSTRUCTIONS. THREE DIFFERENT VIEWS OF DESARGUES' THEOREM

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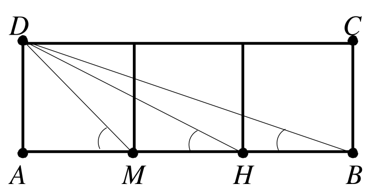
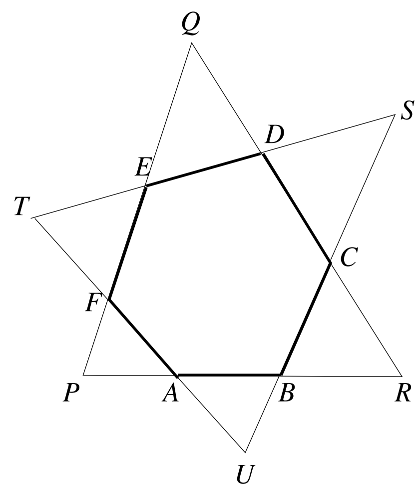
Problem 1. (For Everyone to Play With) Three congruent squares with bases AM, MH and HB, are put next to each other to form a rectangle $ABCD$ (see Fig.1). Show that [1]

$$\angle AMD + \angle AHD + \angle ABD = 90^\circ.$$



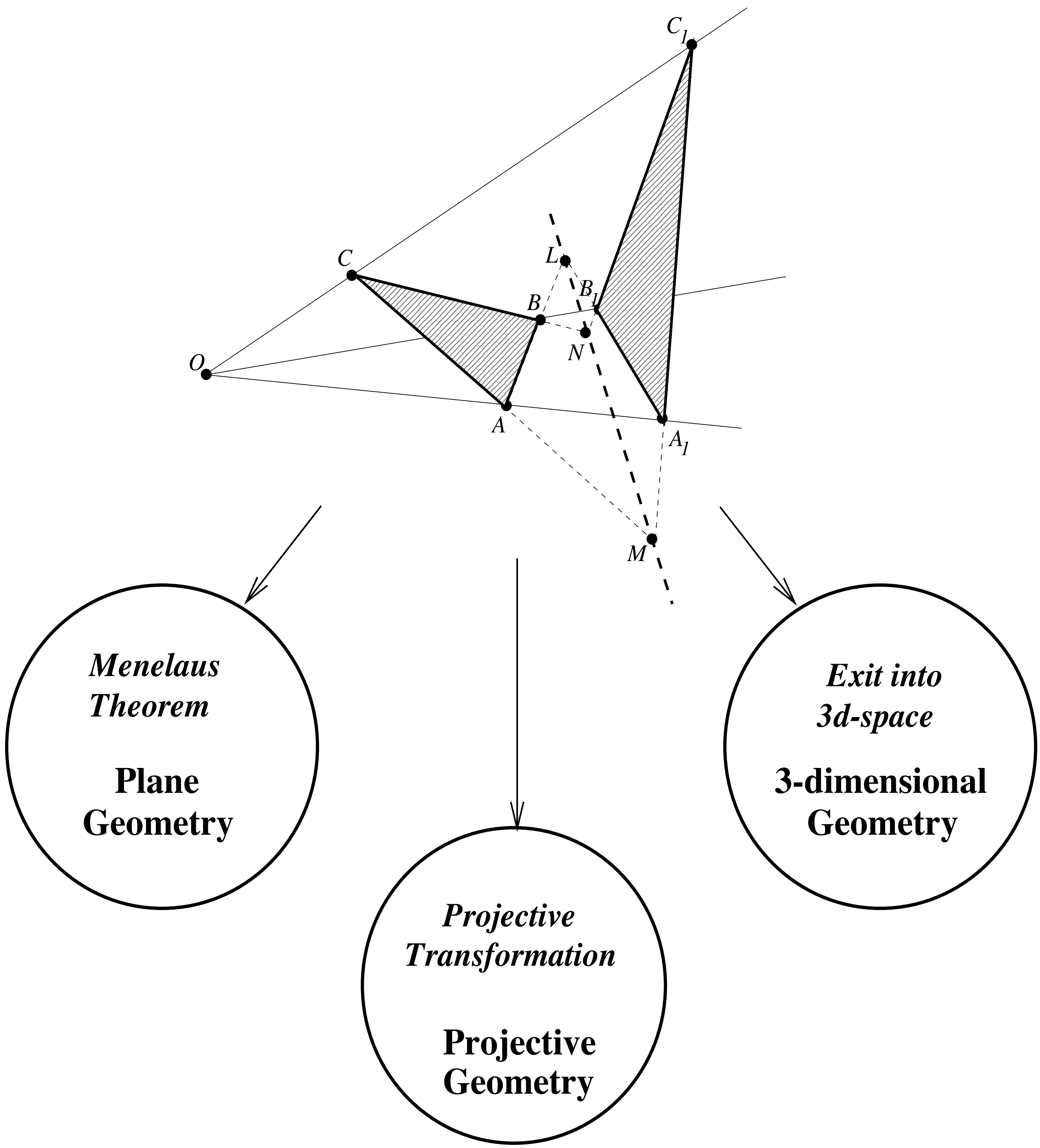


Problem 2. (For the Die-Hards) Let $ABCDEF$ be a convex hexagon. Let P, Q, and R be the intersections of the lines AB and EF, EF and CD, CD and AB, respectively. Let S, T, U be the intersections of the lines BC and DE, DE and FA, FA and BC, respectively. Show that if $AB/PR = CD/RQ = EF/QP$, then $BC/US = DE/ST = FA/TU$. (Math Olympiad Summer Program'98, Homework Assignment.) [2]

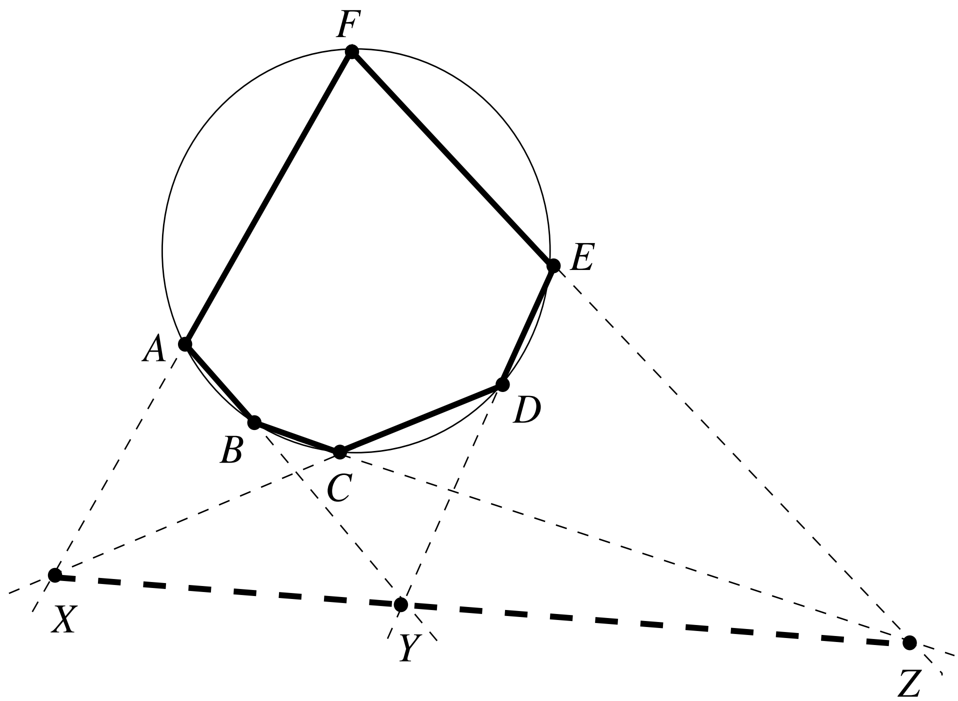
[1] **Note:** The Problem was discussed at length at the Math Circle Workshop on June 6th 1998 in Lawrence Hall of Science. Let's see if anyone remembers the beautiful solution we saw there! Now, imagine you are in 7–8th grade, and you haven't yet heard of "trigonometry" (oops, that's a hint for the advanced! :-)), and your whole world of geometric tricks consists of similar and congruent triangles, and, say, you know that the sum of angles in a triangle is 180°. Can you do with? Play with it and see how far you can get. Solutions to this Problem 1 and Problem 2 will be handed after the talk.

[2] **Warning:** Don't try this at home unless you really know what you are doing! This problem is really hard. When you see the solution you will be surprised that it doesn't require any advanced mathematical tools; BUT how one can come up with such a solution – that where the mystery is! So, good luck. :-)

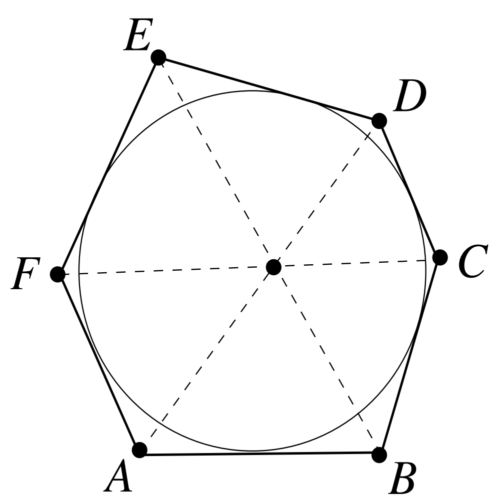
Problem 3. (Desargues' Theorem) $\triangle ABC$ and $\triangle A_1B_1C_1$ are positioned in such a way that lines AA_1, BB_1, and CC_1 intersect in a point O. If lines AB and A_1B_1, AC and A_1C_1, BC and B_1C_1 are pairwise not parallel, prove that their points of intersection, L, M and N, are collinear.

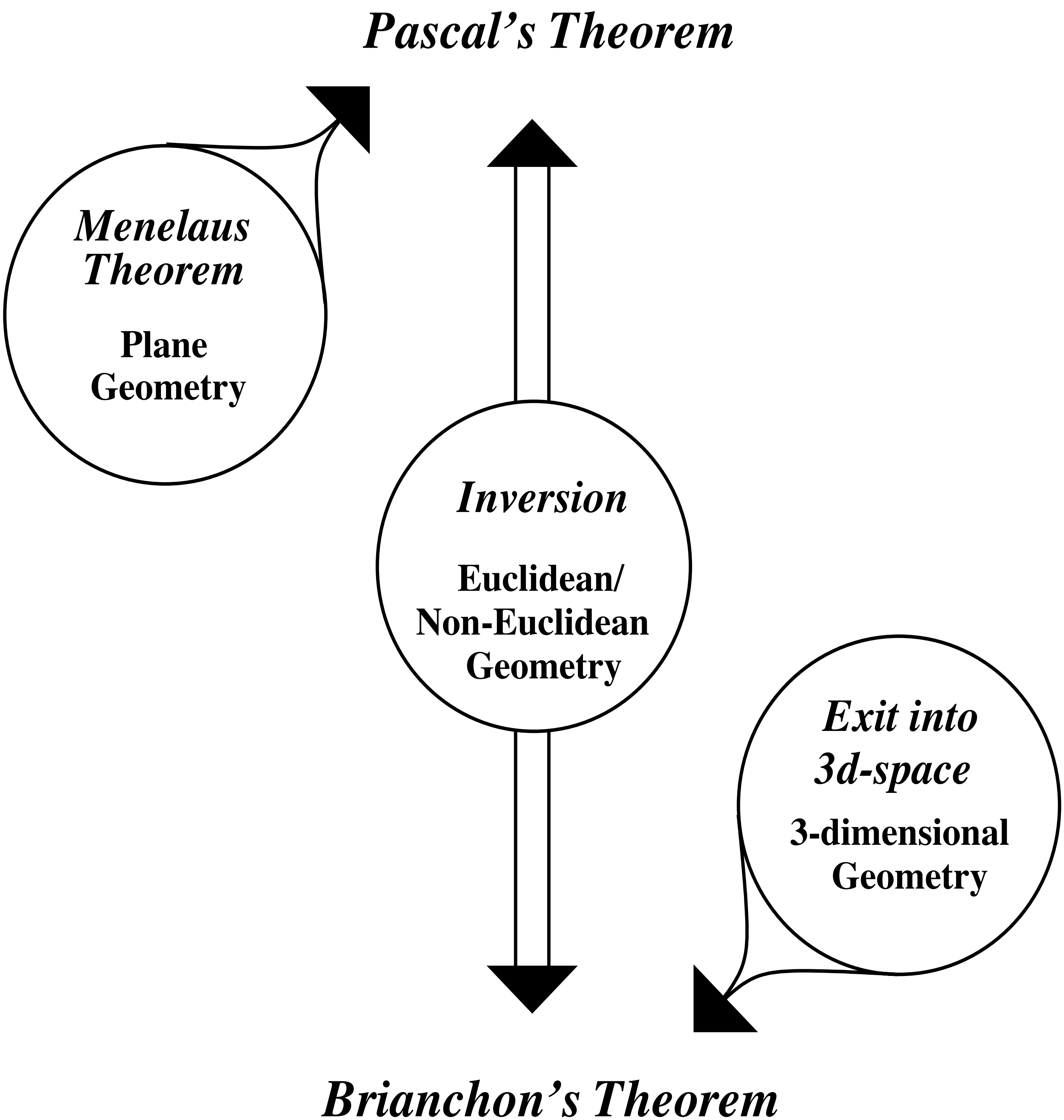


Problem 4. (Pascal's Theorem) If the hexagon $ABCDEF$ is cyclic and its opposite sides, AB and DE, BC and EF, CD and FA, are pairwise not parallel, prove that their three points of intersection, X, Y and Z, are collinear.



Problem 5. (Brianchon's Theorem) If the hexagon $ABCDEF$ is circumscribed around a circle, prove that its three diagonals AD, BE and CF are concurrent.

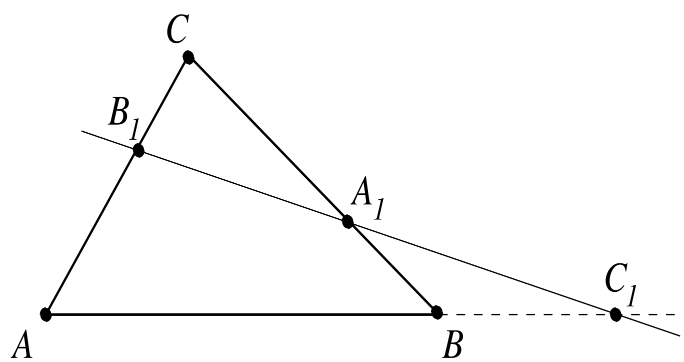



Pascal's Theorem
Menelaus Theorem
Plane Geometry
Inversion
Euclidean/ Non-Euclidean Geometry
Exit into 3d-space
3-dimensional Geometry
Brianchon's Theorem

Note: "We say that several points are *collinear* if they lie on a line. Similarly, several points are *concyclic* if they lie on a circle; an *inscribed* (cyclic) polygon has its vertices lying on a circle. If three distinct points A, B and C are collinear, then the *directed ratio* $\overline{AB}/\overline{CB}$ is the ratio of the lengths of segments AB and CB, taken with a sign "+" if the segments have the same direction (i.e. B is *not* between A and C), and with a sign "−" if the segments have opposite directions (i.e. B is between A and C). Several objects (lines, circles, etc.) are *concurrent* if they all intersect in some point.

Problem 6. (Menelaus' Theorem) Let A_1, B_1 and C_1 be three points on the sides BC, CA and AB of $\triangle ABC$. Then they are collinear if and only if

$$\frac{\overline{AB_1}}{\overline{CB_1}} \cdot \frac{\overline{CA_1}}{\overline{BA_1}} \cdot \frac{\overline{BC_1}}{\overline{AC_1}} = 1.$$



FIRST PROOF OF DESARGUE'S THEOREM VIA MENELAUS

Apply Menelaus' Theorem 3 times to, respectively, $\triangle OBC$ and line NB_1C_1, $\triangle OAB$ and line LB_1A_1, and $\triangle OAC$ and line MA_1C_1:

$$\begin{aligned} \frac{\overline{CN}}{\overline{BN}} \cdot \frac{\overline{BB_1}}{\overline{OB_1}} \cdot \frac{\overline{OC_1}}{\overline{CC_1}} &= 1 \\ \frac{\overline{BL}}{\overline{AL}} \cdot \frac{\overline{AA_1}}{\overline{OA_1}} \cdot \frac{\overline{OB_1}}{\overline{BB_1}} &= 1 \\ \frac{\overline{AM}}{\overline{CM}} \cdot \frac{\overline{CC_1}}{\overline{OC_1}} \cdot \frac{\overline{OA_1}}{\overline{AA_1}} &= 1 \end{aligned}$$

Now we multiply the three equalities and cancel out everything we can. We are left with

$$\frac{\overline{AM}}{\overline{CM}} \cdot \frac{\overline{CN}}{\overline{BN}} \cdot \frac{\overline{BL}}{\overline{AL}} = 1$$

which again by Menelaus (the reverse direction of the theorem) implies that points M, N and N are collinear. □

Question: What happens if some of the pairs of lines in the problem (or in the solution) do not intersect, i.e. they are parallel? Can you still solve the problem using a modification of the above method?

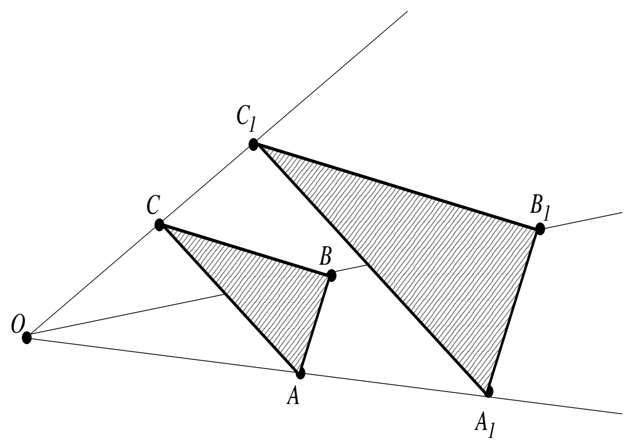
Second Proof of Desargue's Theorem via Projective Geometry

It turns out that it is not so bad to have some of the pairs of lines in the setting of Desargues' be parallel. In fact, making *all* such lines parallel pairwise is the basis for the *Projective Geometry* proof.

There are certain nice transformations in the plane, called *projective*, which send lines to lines – nothing really surprising here: say, reflections across a point or across a line, rotations and parallel translations are examples of such transformations. However, the "magic" of projective transformations works when we are able to "separate" intersecting lines, i.e. making them parallel without changing too much the structure of the original picture. This is possible because we add one extra "line" to the usual plane, called the *line at infinity.* For every family of parallel lines in the usual plane there is a (different) point on the line at infinity l. Conversely, any point on l is "born" by a (unique) family of parallel lines.

Note that it is very hard to imagine exactly the picture of this augmented plane, called the *projective plane.* This is because we are used to think in 3 dimensions, and the projective plane is simply too complex to be "fitted" in 3d–space. Instead of trying to imagine it, think of the projective plane as an *abstract construction* with some useful applications. When you take an introductory course in algebraic geometry, you will see various descriptions of the projective plane. These will hopefully help you construct a satisfactory mental image of the projective plane.

But for now, let just glimpse at the magic performed by a well–chosen projective transformation. In the setting of Desargue's theorem, consider points L and N. If they exist, it means that the pairs of corresponding lines intersect, i.e. $AB \cap A_1B_1 = L$ and $BC \cap B_1C_1 = N$. The idea is to apply a projective transformation to the plane, sending points L and N to the line l at infinity, and thus, in effect making line AB parallel to A_1B_1 (they will intersect at a point "at infinity"), and similarly BC parallel to B_1C_1.



It is now not hard to prove that AC and A_1C_1 are also parallel: use similar triangles $\triangle OAB \sim \triangle OA_1B_1$ (why?), and $\triangle OBC \sim \triangle OB_1C_1$ (why?), to conclude that $OA/OA_1 = OB/OB_1 = OC/OC_1$. This in its turn implies that $\triangle OCA \sim \triangle OC_1A_1$ (why?), and therefore AC is parallel to A_1C_1 (why?). I leave the justifications of "why?"'s to the dedicated reader.

So what? The fact that AC and A_1C_1 are parallel means that they intersect at a point of infinity, namely, M. The nice thing about the projective plane is that no matter what point of view you choose on it, the picture you will see will be essentially the same – you will see the usual (called "finite") plane, and whichever line you won't see, that you can think of as the "line at infinity". In particular, all lines are "created" equal, regardless of whether they are usual lines or the "line at infinity". In other words, the fact that all three points L, M and N happen to lie on the "line at infinity" makes them *collinear.*

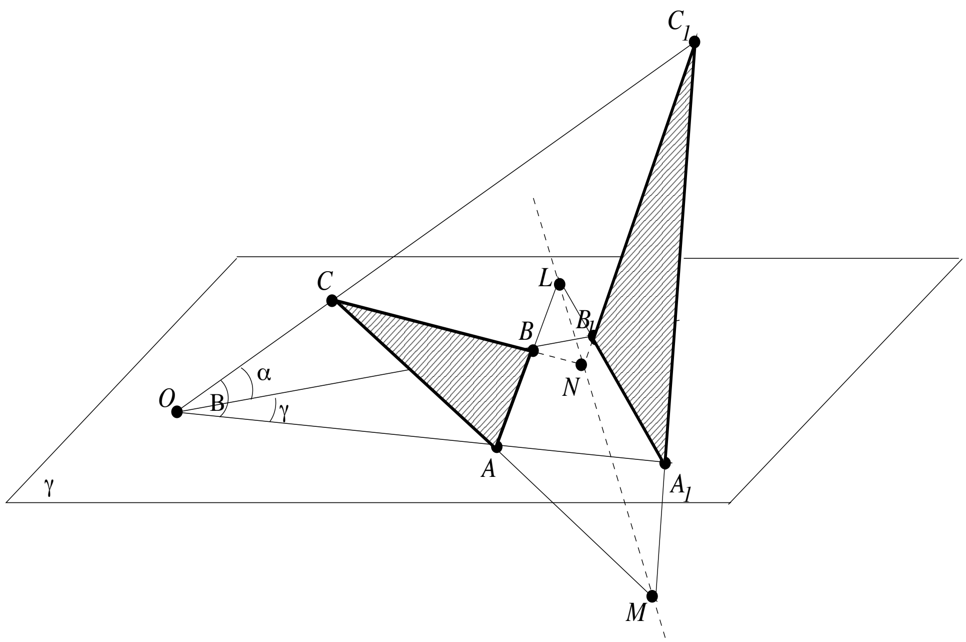
To finish the proof, one has to apply the inverse of whatever projective transformation was applied in the beginning in order to obtain the original picture of Desargues' setting. In the process, the "line at infinity" LMN will be sent to some line in the usual plane, on which our original points L, M and N must have been lying on. □

Third Proof of Desargue's Theorem via Exit into 3d–space

It is very counterintuitive to attempt to solve a (plane) 2d–problem by a 3d–solution. That is, to cook up an argument in 3d–space which somehow implies our 2d–version. This type of reasoning is called *Exit into 3d.*

In the setting of Desargues' theorem, imagine that everything originally lines in some plane γ, but we "lift" the ray OCC_1 vertically from the plane in 3d–space, keeping all lines, triangles and intersection points the same as before. The goal is then to show that the "new" points L, M and N lie on a line in 3d–space; we then project our new 3d–picture back to the original 2d–picture in the plane γ, and necessarily the "space" line $l = LMN$ will project onto another line l_1 in γ. This line γ, we conclude, must have contained our original points L, M and N, so we will be done.

So, what are we waiting for? The 3d–picture looks as follows:



Note that we have created the three planes $\gamma = (OAB)$, $\alpha = (OBC)$, $\beta = (OCA)$, which can be thought of forming part of the pyramid $OABC$ at point, and the two planes formed by the two new triangles: plane $P = (ABC)$ and plane $P_1 = (A_1B_1C_1)$.

Then point L is the intersection of lines AB and A_1B_1; but line AB is the intersection of planes P and γ, while line A_1B_1 is the intersection of planes P_1 and γ. In short:

$$L = AB \cap A_1B_1 = (P \cap \gamma) \cap (P_1 \cap \gamma) = P \cap P_1 \cap \gamma.$$

The serious reader will also verify similarly that

$$M = P \cap P_1 \cap \beta, \text{ and } N = P \cap P_1 \cap \alpha.$$

But planes P and P_1 intersect in some line (why?), which we call on purpose l. Thus, we have seen above that all three points L, M and N, lie on the line $l = P \cap P_1$, i.e. they are collinear. Projecting line l onto the original plane γ yields the wanted line. □

Proof of Pascal's Theorem via Menelaus

Create $\triangle PQR$ by intersecting the following lines: $AB \cap CD = \{R\}$, $CD \cap EF = \{P\}$ and $EF \cap AB = \{Q\}$. Then apply Menelaus' Theorem 3 times to $\triangle PQR$ and lines XAF, CBZ and DYE, respectively:

$$\frac{\overline{PX}}{\overline{RX}} \cdot \frac{\overline{RA}}{\overline{QA}} \cdot \frac{\overline{QF}}{\overline{PF}} = 1$$

$$\frac{\overline{PC}}{\overline{RC}} \cdot \frac{\overline{RB}}{\overline{QB}} \cdot \frac{\overline{QZ}}{\overline{PZ}} = 1$$

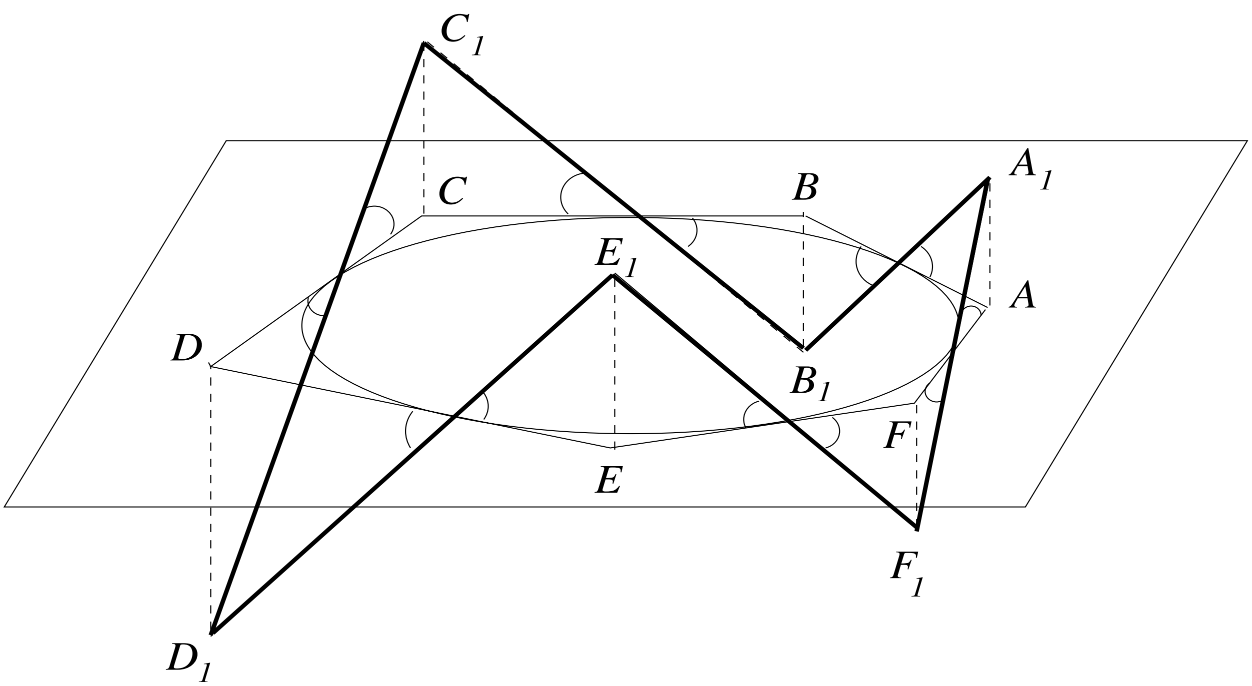
$$\frac{\overline{PD}}{\overline{RD}} \cdot \frac{\overline{RY}}{\overline{QY}} \cdot \frac{\overline{QE}}{\overline{PE}} = 1$$

Now we multiply the three equalities and cancel out everything we can. In particular, note that $RA \cdot RB = RC \cdot RD$, $QF \cdot QE = QA \cdot QB$ and $PC \cdot PD = PF \cdot PE$, by the *Power of Point Theorem* applied consecutively to points R, Q and P, and circle k. Thus, we are left with

$$\frac{\overline{PX}}{\overline{RX}} \cdot \frac{\overline{RY}}{\overline{QY}} \cdot \frac{\overline{QZ}}{\overline{PZ}} = 1,$$

which again by Menelaus (the reverse direction of the theorem) implies that points X, Y and Z are collinear. □

Proof of Brianchon's Theorem via Exit into 3D



• Create a spacial hexagon $A_1B_1C_1D_1E_1F_1$ which projects onto the given planar hexagon, as shown in the picture. (Why does such a hexagon exist? Start with point A_1 in space, projecting onto A, and then construct the remaining 5 points one by one; use six pairs of similar triangles to prove that you will eventually come back to A_1 in your construction.)

• Note that to prove that diagonals AD, BE and CF meet in a point, it will suffice to show that A_1D_1, B_1E_1 and C_1F_1 meet in a point X_1 (in space) – projecting X_1 onto the plane will yield the required intersection point of the original diagonals.

• To show that A_1D_1, B_1E_1 and C_1F_1 intersect in space, it suffices to show that every two of them intersect in space. Indeed, if X, Y and Z are the pairwise intersection of the three segments, AND we suppose by contradiction that X, Y and Z are distinct, this implies that A_1D_1, B_1E_1 and C_1F_1 all lie in a plane (together with X, Y and Z). Now that's a contradiction since $A_1B_1C_1D_1E_1F_1$ is not planar, but spacial by construction.

• To show that, say, A_1D_1 and B_1E_1 intersect in space, it is sufficient to show that lines A_1B_1 and D_1E_1 lie in a plane (why?), or equivalently, to show that A_1B_1 and D_1E_1 intersect.

• Show that all of the 12 marked angles are equal. (Use again the 12 triangles as above, and "equal tangents" from a point to a circle.) This means that all six lines formed by the sides of the spacial hexagon $A_1B_1C_1D_1E_1F_1$ form the same angle with the original plane.

• Show that, say, lines A_1B_1 and D_1E_1 intersect by using two facts: they form the same angle with the original plane, and equal tangent are obtained after extending DE and AB until they meet. (A_1B_1 and D_1E_1 will be parallel if DE and AB are parallel.)

• Put together all pieces above to conclude that the diagonals of the spacial hexagon are concurrent, and hence the diagonals of the original planar hexagon are also concurrent. □

References

1. *"Insights to Geometric Puzzles and Constructions"*, Zvezdelina Stankova-Frenkel, Workshop on Math Circles, Lawrence Hall of Science, June 1998.
2. *"Classical Theorems in Plane Geometry"*, Zvezdelina Stankova-Frenkel, Berkeley Math Circle, September 1999.
3. *"Inversion in the Plane. Part I and Part II"*, Zvezdelina Stankova-Frenkel, Berkeley Math Circle, September/October 1998.
4. *"Introduction to Algebraic Geometry. Projective Geometry"*, Zvezdelina Stankova-Frenkel, lecture notes, Mathematical Olympiad Summer Program, Lincoln, Nebraska, 1999.
5. *"Mathematical Circles (Russian Experience)"*, Fomin, Genkin and Itenberg, American Mathematical Society, 1996.
6. *"Kvant Selecta: Algebra and Analysis, I and II"*, American Mathematical Society, 1999.
7. *"Math Olympiad Contest Problems for Elementary and Middle Schools"*, George Lenchner, Glenwood Publications.
8. *"Euclidean and Non-Euclidean Geometries"*, Marvin Greenberg, W.H. Freeman and Company, 1997.
9. *"Companion to Euclid"*, Robin Hartshorne, American Mathematical Society.

To view items 1–3, see the Berkeley Math Circle webpage at

http://www.math.berkeley.edu/ stankova/

To buy items 5–6 at discount prices, contact Tom Rike at trike@ousd.k12.ca.us

BAY AREA MATHEMATICAL ADVENTURE

GEOMETRIC PUZZLES AND CONSTRUCTIONS.

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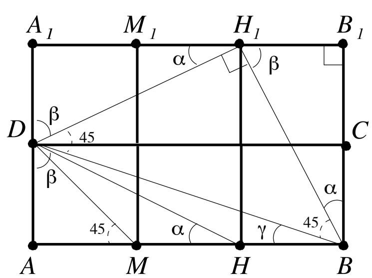
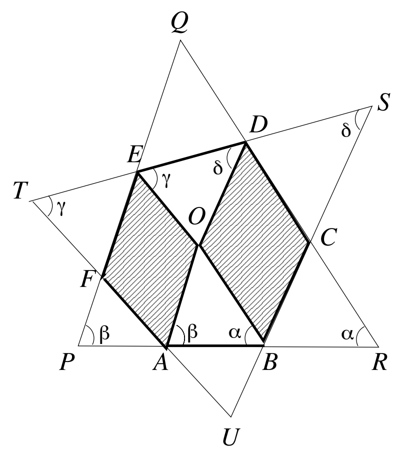
Solutions to Problems 1 and 2

Problem 1. (For everyone to Play With) Three congruent squares with bases AM, MH and HB, are put next to each other to form a rectangle $ABCD$ (see Fig.1). Show that

$$\angle AMD + \angle AHD + \angle ABD = 90^\circ.$$

Solution: Reflect the figure across line DC, i.e. draw three more squares as shown on Fig. 2 below. Connect the new point H_1 with points D and B. Note that $\triangle AHD$, $\triangle A_1H_1D$ and $\triangle B_1BH_1$ are congruent to each other. (Why? They are all right angles with legs of same lengths.) This allows us to call three angles on the picture by α, and three other by β. In particular, $\angle AHD = \alpha = \angle B_1BH_1$.





Further, note that $\triangle DBH_1$ is a right isosceles triangle. Indeed, $|DH_1| = |BH_1|$ (Why? Because of two of the above triangles are congruent. Which triangles do I have in mind?), and

$$\angle DH_1B = 180^\circ - \angle DH_1A_1 - \angle B_1H_1B = 180^\circ - \alpha - \beta = 90^\circ.$$

The last follows from the fact that α and β are the two acute angles in right $\triangle AHD$, and hence they sum up to 90°. Thus, in $\triangle DBH_1$, $\angle DBH_1 = 45^\circ$.

To finish the proof, note that the three wanted angles are $\angle AMD = 45^\circ$ (Why? $\triangle AMD$ is also right isosceles,) $\angle AHD = \alpha$ and $\angle ABD = \gamma$; and that they appear "miraculously", in the right $\angle ABB_1$:

$$90^\circ = \angle ABB_1 = \gamma + 45^\circ + \alpha. \qquad \square$$

Now, everyone understands that the construction in the above solution is really a very original idea, and there is no guarantee that everyone (or anyone!) will come up with this same idea within a "finite amount of time", as mathematicians like to say. Thus, instead we propose here a simple trigonometric solution, which doesn't require any original thinking, but has the drawback of giving us no idea of *why* these angles sum up to 90°. Enjoy it nevertheless.

Alternative Solution: Since $\angle DMA = 45^\circ$ (from right isosceles $\triangle AND$ as above,) it suffices to show that $\angle AHD + \angle ABD = 45^\circ$. Name these two angles by α and γ as above. Since they are both acute angles, they cannot sum up to more than 180°; thus, if we show that $\tan(\alpha+\gamma) = 1$, we will be able to conclude that $\alpha + \gamma = 45^\circ$.

The formula for the tangent of a sum comes to the rescue:

$$\tan(\alpha+\gamma) = \frac{\tan\alpha + \tan\gamma}{1 - \tan\alpha \cdot \tan\gamma}.$$

From $\triangle AHD$ and $\triangle ABD$, respectively, we find $\tan\alpha = |AD|/|AH| = 1/2$ and $\tan\gamma = |AD|/|AB| = 1/3$. Substituting 1/2 and 1/3 into the above tangent formula yields

$$\tan(\alpha+\gamma) = \frac{1/2 + 1/3}{1 - 1/2 \cdot 1/3},$$

which I leave to the diligent reader to check that it equals 1. □

Question: Why did I use tangents? Would it be easier to use sines or cosines, or some other trigonometric function of the angles?

Problem 2. (For the Die-Hards) Let $ABCDEF$ be a convex hexagon. Let P, Q, and R be the intersections of the lines AB and EF, EF and CD, CD and AB, respectively. Let S, T, U be the intersections of the lines BC and DE, DE and FA, FA and BC, respectively. Show that if $AB/PR = CD/RQ = EF/QP$, then $BC/US = DE/ST = FA/TU$. (Math Olympiad Summer Program'98, Homework Assignment.)

Solution: The given triple ratios remind us suspiciously of a criterion for similar triangles (SSS). It is as if someone wants to tell us that $\triangle PRQ$ is similar to another triangle with sides AB, CD and EF, but no such similar triangle can be found on the given picture. So, let's construct it!

Draw a line through A parallel to PQ, and another line through B parallel to RQ, and let them intersect in point O (Would they intersect? Why?) Connect O with E and with D. Our goal is to prove that $AOEF$ and $BCDO$ are both parallelograms, and use this to prove what is wanted in the problem, but let's not get ahead of ourselves, and let's do everything step by step.

For starters, do you see any similar triangles? By construction, $\triangle ABO$ and $\triangle PQR$ are similar: check out their equal angles α's and β's from the parallel lines in our construction. Therefore, the sides of these two triangles are proportionate, i.e.

$$AB/PR = BO/RQ = OA/QP.$$

But we have by hypothesis that

$$AB/PR = CD/RQ = EF/QP.$$

Since the first ratio is the same in both equations, all those five ratios are equal, in particular, $BO/RQ = CD/RQ$ and $OA/QP = EF/QP$. We conclude that $BO = CD$ and $OA = EF$.

Recall now that by construction BO is parallel to CD, and OA is parallel to EF. Therefore, indeed we do have parallelograms $AOEF$ and $BCDO$.

Now, we can play the same game for $\triangle TSU$ and $\triangle EOD$, by reversing the above argument. Are they similar? Since EO and TU are parallel, and DO and SU are parallel (from the parallelograms above) we conclude that the two triangles have equal angles γ's and δ's, and therefore they are indeed similar.

Thus, the sides of $\triangle TSU$ and $\triangle EOD$ are proportionate:

$$OD/US = DE/ST = OE/TU.$$

But $OD = BC$ and $OE = FA$ (again from the parallelograms), thus

$$BC/US = DE/ST = FA/TU. \qquad \square$$