BERKELEY MATH CIRCLE MONTHLY CONTEST #6, DUE MARCH 26, 2000

1. Let $g_1(x)$, $g_2(x)$, ... , $g_5(x)$ be polynomials with integer coefficients. Suppose that their product $f(x) = g_1(x)g_2(x)g_3(x)g_4(x)g_5(x)$ satisfies $f(1999) = 2000$. Prove that for some $i \in \{1, 2, 3, 4, 5\}$, the sum of the coefficients of $g_i(x)$ is odd.
2. Let d be a fixed positive integer. Prove that there exists a unique polynomial $S(n)$ such for every integer $n \geq 0$,
$$S(n) = \sum_{k=0}^{n} k^d = 0^d + 1^d + \cdots + n^d.$$
(Hint: solve for the coefficients of a polynomial $S(n)$ that satisfies $S(n) - S(n-1) =$ something.) Also prove that $S(n)$ can be expressed in the form
$$c_0 + c_1(2n+1) + c_2(2n+1)^2 + \cdots + c_{d+1}(2n+1)^{d+1},$$
where the c_i are rational numbers such that $c_i c_{i+1} = 0$ for $i = 0, 1, \ldots, d$.
3. On the sides of a convex quadrilateral $ABCD$, construct squares externally. Prove that the quadrilateral with vertices at the centers of the squares has equal and perpendicular diagonals.
4. Let $P(x)$ be a polynomial with integer coefficients. Let $a_0 = 0$ and for $i \geq 0$ define $a_{i+1} = P(a_i)$. Prove that $\gcd(a_m, a_n) = a_{\gcd(m,n)}$ for any integers $m, n \geq 1$.
5. Acute triangle ABC is made of solid metal, and it is on top of a wooden table. Points P on AB and Q on AC are such that the perpendicular to AB through P intersects the perpendicular to AC through Q *inside* the triangle. Nails are hammered into the table at P and Q. (The nails do not go through the triangle, but are at its edges, bounding the triangle.) Prove that there is a *unique* position R on BC such that if a nail is hammered into the table at R, the triangle will no longer be able to move (within the plane of the table).

Please write solutions to different problems on separate pages. At the top of each page, write your name, school, city, contest number, and problem number. Remember that these problems are not to be discussed with anyone until after the due date. Please go to `http://www.math.berkeley.edu/~stankova/MathCircle/Joyce/index2.html` for more information about the contest.

Date: February 20, 2000.