BERKELEY MATH CIRCLE MONTHLY CONTEST #4, DUE JANUARY 23, 2000

1. Do there exist positive integers x, y, and z such that $28x + 30y + 31z = 365$? What is the largest positive integer n such that the equation $28x + 30y + 31z = n$ *cannot* be solved in positive integers x, y, and z? (Explain why your answer is correct.)
2. Let $x_1 = x_2 = 1$ and $x_{n+2} = 2x_{n+1} + 8x_n - 1$ for $n \geq 1$. Prove that x_n is a square for all n.
3. Circles k_1, k_2, k_3 intersect as follows: $k_1 \cap k_2 = \{A, D\}$, $k_1 \cap k_3 = \{B, E\}$, and $k_2 \cap k_3 = \{C, F\}$. Also, $ABCDEF$ is a non-selfintersecting hexagon (possibly non-convex). Prove that
$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.$$
(Hint: first prove that AD, CF, BE meet at some point O.)
4. Let p be an odd prime and a, b positive integers satisfying $(p+1)^a - p^b = 1$. Show that $a = b = 1$.
5. Equilateral triangles ABC_1, BCA_1, CAB_1 are constructed outwards on the sides of a triangle ABC. Prove
   (a) The centers of these three equilateral triangles form another equilateral triangle.
   (b) The segments AA_1, BB_1, CC_1 are concurrent.
   (c) The segments AA_1, BB_1, CC_1 have equal lengths.

Please write solutions to different problems on separate pages. At the top of each page, write your name, school, city, contest number, and problem number. Remember that these problems are not to be discussed with anyone until after the due date. Please go to `http://www.math.berkeley.edu/~stankova/MathCircle/Joyce/index2.html` for more information about the contest.

Date: December 12, 1999.