Berkeley Math Circle
Take-Home Contest #1 – Solutions

1. Let k and n be positive integers such that $k < 2^{n+1} - 1$. Prove that there is a sum of *exactly* n powers of 2 that is divisible by k. (Example: if $k = 9$ and $n = 4$, then $2 + 4 + 16 + 32$ is divisible by 9.)

   Solution: Write k in base 2 and suppose there are r 1's. Since $2^{n+1} - 1$ is the smallest number with at least $n + 1$ 1's, we conclude $r \leq n$. Also k clearly divides a sum of r powers of 2, since k is such a sum. Now we claim that if k divides a sum of i powers of 2, it also divides a sum of $i + 1$ such powers. To see this, suppose that 2^l is one of our i powers with $l > 0$; then rewrite this as $2^{l-1} + 2^{l-1}$ and keep the remaining $i - 1$ powers the same. This gives us the needed sum of 2^{i+1} powers. We can always do this unless all i of our powers are equal to 2^0; in this case we have $k \mid i$, but then $k \mid 2i$, so we can multiply everything by 2 and then split one of the resulting 2^1's as described previously.

   Now by induction, it follows that k divides a sum of i powers of 2 for all $i \geq r$. In particular, since $n \geq r$, we have what we need.

   Remark: In fact, one can show that there are n *distinct* powers of 2 whose sum is divisible by k. It follows from Euler's theorem that $2^{l+n+i\phi(k)}$ is congruent to $2^n \cdot 2^l$ modulo k for any positive integer i, where $\phi(k)$ is the number of positive integers $\leq k$ and relatively prime to k. We can thus replace each original exponent l with values of $l + n + i\phi(k)$, where i's can be chosen so that all the exponents are distinct, and we obtain another number divisible by k, since it is congruent to 2^n times the original number modulo k.

2. Ten tourists visiting a tropical island are captured by one of the natives, who happens to be a cannibal. The cannibal explains to them that it is the custom on his island to give prisoners a test before eating them. Tomorrow he will line them up in single file, and he will place a black or white hat on each of them, so that each tourist can only see the colors of the hats of the tourists ahead in the line. Then, starting from the rear of the line, the cannibal will ask each tourist to guess the color of his hat, based on the colors of the hats ahead, and based on any earlier guesses that were made by tourists behind him. (The tourists are not allowed to signal each other in any other way.) If a tourist names the wrong color, the cannibal silently eats that tourist before moving on to the next. At the end, any uneaten tourists are freed. Show that if the tourists are allowed to plan a strategy the night before they are tested, they can guarantee that at least nine of them will escape.

   Solution: The "martyr" tourist in the back counts the number of black hats he sees. If the number is odd, he guesses "black"; if it is even, he guesses "white." We claim that this will enable all remaining "lucky" tourists to identify their hats.

   The second tourist from the back can see all the hats that the martyr counted except her own. She observes whether she sees an odd number or an even number of black hats and compares this to what the martyr saw (and reported). If the parities are the same, her hat must be white; if they are opposite, her hat must be black. Thus she can correctly state her color.

   The next tourist now knows the colors of the 7 remaining hats ahead, as well as the hat behind him (since he heard the previous guess which was guaranteed to be correct). So he can compare this with the martyr's observation, using the same parity reasoning as before, and thus correctly deduce the color of his hat. This method continues inductively through the line: each lucky tourist, on his turn, knows all the colors of the other lucky tourists' hats, either from seeing them or from hearing the preceding guesses; this tourist can apply the parity argument and thus also identify the color of his hat. So every lucky tourist can guess correctly, as claimed.

   Remark: One can also show that (up to the trivial interchanging of "black" and "white") this is the only strategy that necessarily saves at least nine tourists.

3. Let n be an integer greater than 2. Positive real numbers x and y satisfy $x^n = x + 1$ and $y^{n+1} = y^3 + 1$. Prove that $x < y$.

Solution: It is clear from the given that $x^n > 1$ and $y^{n+1} > 1$; therefore $x > 1, y > 1$. From this we get $0 < (y-1)(y^2-1) = y^3 - y^2 - y + 1 \Rightarrow y^2 + y < y^3 + 1 = y^{n+1}$ and, dividing by y, we obtain $y + 1 < y^n$. Thus $y^n - y > 1 = x^n - x$. Now if $y \le x$ then we also have $0 < y^{n-1} - 1 \le x^{n-1} - 1$ and so $y^n - y = y(y^{n-1} - 1) \le x(x^{n-1} - 1) = x^n - x$, which is a contradiction; hence we must have $y > x$ as claimed.

4. Let $z = \cos 2\pi/n + i \sin 2\pi/n$ where n is a positive odd integer. Prove that

$$\frac{1}{1+z} + \frac{1}{1+z^2} + \frac{1}{1+z^3} + \cdots + \frac{1}{1+z^n} = \frac{n}{2}.$$

Solution: Consider the polynomial $P(x) = x^n + (x-1)^n$. We claim that for each integer i, $x = 1/(1+z^i)$ is a root. Indeed, we have $(-z^i)^n = (-1)^n z^{in} = -1$ (since $z^n = 1$ and n is odd) and so

$$x^n = \frac{1}{(1+z^i)^n}; \quad (x-1)^n = \left(\frac{-z^i}{1+z^i}\right)^n = \frac{-1}{(1+z^i)^n} \Rightarrow x^n + (x-1)^n = 0.$$

Moreover, since the values of z^i are distinct for $i = 1, 2, \ldots, n$, we have found n different roots of this polynomial which has degree n, so by the Fundamental Theorem of Algebra, they are all the roots and we can write

$$P(x) = c\left(x - \frac{1}{1+z}\right)\left(x - \frac{1}{1+z^2}\right)\cdots\left(x - \frac{1}{1+z^n}\right) \text{ for some constant } c.$$

If we expand the expression on the right side, the coefficient of x^n is c and the coefficient of x^{n-1} is $-c[1/(1+z) + 1/(1+z^2) + \cdots + 1/(1+z^n)]$. On the other hand, on the left side, the coefficient of x^n is 2 and the coefficient of x^{n-1} is $-n$ (found by expanding $(x-1)^n$ using the binomial theorem). Equating coefficients, we get $c = 2$ and thus $-2[1/(1+z) + 1/(1+z^2) + \cdots + 1/(1+z^n)] = -n$ which yields what we need.

Remark: There are many ways of solving this problem. Perhaps the simplest is to note that the last term is 1/2 and the others can be arranged into pairs each having sum 1. Also, two contestants used the identity $1/(1+z^i) = (1 - z^i + z^{2i} - \cdots + z^{(n-1)i})/2$ and then summed the resulting numerator terms. There is even a calculus solution: write the derivative of $x^n + 1$ at $x = 1$ in two different ways and compare the two expressions.

5. An apple is in the shape of a ball of radius 31 mm. A worm gets into the apple and digs a tunnel of total length 61 mm, and then leaves the apple. (The tunnel need not be a straight line.) Prove that one can cut the apple with a straight slice through the center so that one of the two halves is not rotten.

Solution: Let A and B be the endpoints of the tunnel, and let C be the point diametrically opposite to A. We claim that the plane π which perpendicularly bisects BC gives the desired cut. Indeed, this plane contains the center of the apple since it is equidistant from B and C; now assume for the sake of contradiction that the tunnel enters both halves of the apple. Then it must cross π at some point R. Reflect that portion of the tunnel lying between R and B across π, thus obtaining a tunnel from R to C. Note that reflection preserves the length of this segment of tunnel. Thus we obtain a tunnel from A to R to C, which has the same length as the original (61 mm). On the other hand, since A and C are diametrically opposite, by the triangle inequality the tunnel has length $\ge AC = 62$ mm. This is a contradiction. Thus our tunnel cannot intersect both halves, which is what we want.