BAMO 1999: SOLUTIONS TO PRACTICE PROBLEMS II

BAMO PROBLEMS COMMITTEE
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Participation in BAMO is already an achievement - four hours thinking over hard math problems is what counts, not who solved how many problems! The problems are arranged in approximately increasing difficulty; in particular, problems 4 and 5 are quite hard. We expect no more than 1 or 2 students (if any) to solve all problems, and few to solve completely 3 problems. Thus, solving even one problem is already a FINE performance! Remember that you are competing against students in your age group – if the problems seem hard (simple) to you, they are probably hard (simple) for the other contestants too. The idea of BAMO is that students find it a fun experience and give it all they have during the contest; and after the contest is over, students continue thinking over the proposed problems.

Problem 1. Prove that $1993^{1993} + 1994^{1994} + 1995^{1995} + 1996^{1996}$ is divisible by 10.

Solution. Since $5^2 = 25$ ends in 5 again, and $6^2 = 36$ ends in 6 again, no matter to what power we raise 5 or 6, the resulting numbers will end in 5 or 6, respectively. Thus, 1995^{1995} ends in 5 and 1996^{1996} ends in 6. Further, $4^1 = 1$, $4^2 = 16$, $4^3 = ...4$, $4^4 = ...6$, etc, i.e. an odd power of 4 ends in 4, and an even power of 4 ends in 6 (why?) Hence, 1994^{1994} ends in 6. So, the last digit of $1994^{1994} + 1995^{1995} + 1996^{1996}$ is the same as the last digit of $6 + 5 + 6 = 17$, i.e. it ends in 7. It remains to show that 1993^{1993} ends in 3 (so that $3 + 7 = 10$.)

Indeed, $3^1 = 3$, $3^2 = 9$, $3^3 = ...7$, $3^4 = ...1$, $3^5 = ...3$, and the pattern will be $3, 9, 7, 1, 3, 9, 7, 1, ...,$ etc. So we need to figure out into which of these slots will the 1993-rd power of 3 fall. Equivalently, we need to find the remainder of 1993 when divided by 4 (the length of the pattern above is 4). Now, $1993 = 4 \cdot 498 + 1$: 3^{1993} will complete 498 cycles of the pattern, and will end in 3. This shows that 1993^{1993} ends in 3. Therefore, the last digit of the total sum is the last digit of $3 + 6 + 5 + 6 = 20$, 0, which makes the sum divisible by 10. □

Problem 2. King Arthur and his 100 knights are having a feast at a round table. Each person has a glass with white or red wine in front of him. Exactly at midnight, every person moves his glass in front his left neighbor if the glass has white wine, or in front of his right neighbor if the glass has red wine. It is known that there is at least one glass with red and one glass with white wine.

(a) Prove that after midnight there will be at least one person without a glass of wine in front of him.
(b) If Lady Guinivera, the Queen, calls the King off the table before midnight, will the conclusion of part (a) still going to be correct?

Solution (a). Suppose not. Label all seats as $1, 2, ..., 101$ by going around the table anticlockwise. If each person still has a glass after midnight, then no one has received glasses from *both* of his neighbors, i.e. any two people with exactly one person between must have the same color wine. Thus, the following seats must correspond to the *same* color wine: $1, 3, 5, ..., 99, 101, 2, 4, ..., 98, 100$. But these are *all* seats – this contradicts the hypothesis that there are glasses with red and glasses with white wine! Therefore, the supposition is wrong, and at least one person will be left without a glass after midnight. $\square$

Solution (b). The conclusion won't be correct. For example, give white wine to all odd seated people $1, 3, 5, ..., 99$, and red wine to all even seated people $2, 4, 6, ..., 100$. After midnight the two groups will switch the colors of their wines. $\square$

Problem 3. On the bases AB and CD of a trapezoid $ABCD$ draw two squares externally to $ABCD$. Let O be the intersection point of the diagonals AC and BD, and let O_1 and O_2 be the centers of the two squares. Prove that O_1, O and O_2 lie on a line (i.e. they are *collinear*; see Figure.)

Solution. The idea is to show that $\triangle OCO_2 \sim \triangle OAO_1$. Indeed, first notice that $\triangle AO_1B \sim \triangle CO_2D$ – both are right isosceles triangles. Therefore, $AO_1 : CO_2 = AB : CD$. But $\triangle AOB \sim \triangle COD$ ($AB||CD \Rightarrow$ all three angles are the same), so $AB : CD = AO : CO$. This implies $AO_1 : CO_2 = AO : CO$. Further, $\angle O_1AO = \angle O_1AB + \angle BAO = 45^o + \angle DCO = \angle OCO_2$, and we finally conclude that $\triangle OCO_2 \sim \triangle OAO_1$.

Hence, $\angle AOO_1 = \angle COO_2$. Since AOC is a line, then O_2OO_1 is also a line. $\square$

Note: Alternatively, apply dilation with center O and ratio $AO : OC$ - it will take one square to the other, and correspondingly, one center to the other.

PROBLEMS 3 AND 5

Problem 4. The real positive numbers $a_1, a_2, ..., a_n, ...$ satisfy the relation $a_{n+1}^2 = a_n + 1$ for all $n = 1, 2, ...$ Prove that at least one of the a_i's must be an irrational number.

Solution. Argue by contradiction. Suppose that for all i, $a_i = p_i/q_i$ for some relatively prime positive integers p_i, q_i. Substitute in the given relation:

$$\frac{p_{n+1}^2}{q_{n+1}^2} = \frac{p_n + q_n}{q_n} \Rightarrow p_{n+1}^2 q_n = q_{n+1}^2 (p_n + q_n).$$

But p_{n+1}^2 and q_{n+1}^2 are relatively prime, and q_n and $q_n + p_n$ are also relatively prime (why?). This means that $p_{n+1}^2 = p_n + q_n$ and $q_n = q_{n+1}^2$. In particular, $q_{n+1} = \sqrt{q_n} = \sqrt[4]{q_{n-1}} = \cdots = \sqrt[2^n]{q_1}$. Unless $q_1 = 1$, we will eventually run out of perfect squares in q_1 and q_n will not be an integer, a contradiction. Thus, the only possibility is $q_1 = 1$, so that all $q_n = 1$, and our numbers a_n are after all integers.

But this is absurd! Indeed, $a_{n+1} = \sqrt{a_n + 1} < a_n$ (because $a_n + 1 < a_n^2$ for all integers $a_n > 1$, and $a_1 = 1$ implies an immediate contradiction for $a_2 = \sqrt{2}$ is irrational,) so that we obtain an *infinite strictly descreasing* sequence of positive integers, and there isn't simply such a thing! This shows that our supposition was wrong and at least one of the a_n's will be irrational. □

Problem 5. [**Simpson's Line**] Let $\triangle ABC$ be inscribed in a circle k, and let M be an arbitrary point on k different from A, B, C. Prove that the feet of the three perpendiculars from M to the sides of $\triangle ABC$ are collinear. (Note: You may have to extend some sides to find these feet: see Figure.)

Solution. It will suffice to show that $\angle AQR = \angle CQP$ (compare with Problem 3.) Note that both quadrilaterals $MQAR$ and $MQPC$ are cyclic: $\angle MRA = 90^o = \angle MQA$, and $\angle MQC = 90^o = \angle MPC$. Using inscribed angles in these quadrilaterals, we obtain that the two angles in questions equal correspondingly to:

$$\angle AQR = \angle AMR, \ \angle CQP = \angle CMR. \qquad (1)$$

Look carefully at $\triangle MAR$ and $\triangle MCP$: both have one right angle, and we hope to show that two more angles are the same. Equivalently, we want to show that these two triangles are similar. Can we do that? Yes. Their third angles are the same: $\angle MCP = \angle MAR$ because $\angle MAR = 180^o - \angle MAB = \angle MCB$ from the inscribed quadrilateral $MCBA$.

Thus, we conclude that $\triangle MAR \sim \triangle MCP$ (two same angles), so their third angles are also the same: $\angle CMP = \angle RMA$. Combining this with (1), we finally obtain $\angle AQR = \angle CQP$, and therefore P, Q, R are collinear. □