BAMO 1999: SOLUTIONS TO PRACTICE PROBLEMS

BAMO PROBLEMS COMMITTEE

Problem 1. Let A and B be two different hospitals that treat exactly the same number of patients during a year. Each patient suffers from one of two diseases, X or Y. Hospital A cures a greater percentage of its patients than hospital B. Is it possible that hospital B cures both a greater percentage of X–patients than A, and a greater percentage of Y–patients than A?

Solution. This is the well-known Simpson's Paradox: just make B specialize in a riskier disease. For example, let B treat 90 cancer patients and 10 acne patients, with respective cure rates of 50% and 100%. Let A treat 10 cancer and 90 acne patients, with cure rates of 0% and 70%, respectively. $\square$

Problem 2. Bildert works in a cubicle in an office which consists of 27 cubicles arranged in a $3 \times 3 \times 3$ cube. Any two cubicles sharing a wall have a connecting door on this wall; for example, the corner cubicles have exactly 3 doors, while the center cubicle has 6 doors: one on each wall, one on the floor, and one on the ceiling. If Bildert starts at the central cubicle, can he visit each of the other 26 cubicles exactly once (i.e. without revisiting any cubicles)?

Solution I. The answer is *no*. Denote the central cubicle by C, and denote the vertex, edge and face cubicles by V, E and F, respectively. The trip must start with C and include every one of the 8 V's, 6 F's, and 12 E's. The sequence must begin with CFE. Each cubicle V is adjacent only to E cubicles, and each F cubicle except for the very first one, is adjacent only to E cubicles. This means that for the remaining 13 V's and F's that follow the initial CFE, at least 12 new E's are needed. This means that we need at least 13 E's, impossible. $\square$

Solution II. Divide the cubicles into two subsets depending on the parity of the sums of coordinates of each cubicle. (Thus, in the above notation, one subset consists of cubicles C and E's, and the other subset consists of F's and V's.) Each move alternates between the two subsets. The starting subset has 13 cubicles, while the other subset has 14 cubicles – obviously we cannot keep alternating, because we'll run short of cubicles in the starting subset. Hence, Bildert *cannot* visit all cubicles without repetitions. $\square$

Problem 3. Let a, b, c, d, e, f be positive integers, each at least 2, whose sum is S. Prove that

$$a(a-1)+b(b-1)+c(c-1)+d(d-1)+e(e-1)+f(f-1) \leq (S-10)(S-11)+10.$$

When is equality achieved?

Solution I. Adding $-3S+24$ to both sides, makes the inequality equivalent to

$$(a-2)^2+(b-2)^2+(c-2)^2+(d-2)^2+(e-2)^2+(f-2)^2 \leq (S-12)^2.$$

Substituting $A = a-2, B = b-2$, etc., this is the same as

$$A^2+B^2+C^2+D^2+E^2+F^2 \leq (A+B+C+D+E+F)^2.$$

On the left side we have the sum of squares of nonnegative numbers and on the right side we have the square of the sum. The latter is always larger except when all pairwise products AB, AC, DF... are zeros. This happens when all but one of A, B, C, D, E, F are zero, correspondingly, when all but one of a, b, c, d, e, f are 2. $\square$

Solution II. The given inequality can be written as

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \leq (S-10)^2 + 20.$$

Now, for any two numbers $x, y \geq z$, we have

$$x^2 + y^2 \leq z^2 + (x+y-z)^2. \tag{1}$$

Indeed, this is equivalent to

$$x^2 - z^2 \leq (x+y-z)^2 - y^2 \Leftrightarrow (x-z)(x+z) \leq (x-z)(x+2y-z) \Leftrightarrow 0 \leq 2(x-z)(y-z),$$

and the last is true because $x \geq z$ and $y \geq z$. Note that (1) replaces the numbers (x, y) by $(z, x+y-z)$ without changing the sum of the two numbers, but increases the sum of their squares. In the original problem, we do this for $a, b \geq 2$: we replace (a, b) by $(2, a+b-2)$:

$$a^2 + b^2 \leq 2^2 + (a+b-2)^2.$$

We then do the same for $(a+b-2, c)$: replace them by $(2, a+b+c-4)$, and so on. In the end, we will have replaced five of the original numbers by 2's, and the last by $a+b+c+d+e+f-10 = S-10$.

$$\Rightarrow a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \leq 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + (S-10)^2 = 20 + (S-10)^2.$$

Equality is achieved if and only if there are equalities each time we apply (1), i.e. five of the given numbers are 2's, and the remaining number is therefore $S - 10$. $\square$

Solution III. We first show the following inequality:

$$x(x-1) + y(y-1) \leq (x+y-2)(x+y-3) + 2, \ x, y \geq 2. \tag{2}$$

Consider two complete graphs, with x and y vertices, respectively. (A complete graph has all possible edges drawn.) Thus we have $\binom{x}{2}$ and $\binom{y}{2}$ edges in the two graphs. If we glue the graphs together on an edge, we produce a new graph with $x+y-2$ vertices. Count edges: the original configuration had $\binom{x}{2} + \binom{y}{2}$ edges, while the new configuration has at most $\binom{x+y-2}{2}$ edges. Since we lost an edge when we glued the two graphs together, we conclude that

$$\binom{x}{2} + \binom{y}{2} \leq \binom{x+y-2}{2} + 1.$$

This is equivalent to (2) after multiplying by 2. Equality is attained if and only if the new graph is also complete, i.e. one of the original graphs must have been just an edge ($x = 2$ or $y = 2$). From here, apply consecutively (2) to the desired inequality. Again, maximum is attained if and only if five of the given numbers are 2's. $\square$

Problem 4. In the $O-E$ game, a round starts with player A paying c cents to player B. Then A secretly arranges the numbers $1, 3, 5, 7, 9, 11, 13$ in some order as a sequence $a_1, a_2, ..., a_7$, and B secretly arranges $2, 4, 6, 8, 10, 12, 14$ as a sequence $b_1, b_2, ..., b_7$. Finally, the players show their sequences and B pays A one cent for each i in $X = \{1, 2, 3, 4, 5, 6, 7\}$ such that $a_i < b_i$. This finishes the round. What number c would make the game fair? (The game is fair if the total payments by A to B equals the total payments by B to A after all possible distinct rounds are played exactly once.)

Solution I. Let k be in X. There are $7!6!(8-k)$ choices of the sequences $a_1, a_2, ..., a_7$ and $b_1, b_2, ..., b_7$ for which $2k-1$ is an a_j with $a_j < b_j$. Indeed, the a's can be any of the 7! permutations of the 7 odd integers; then j is the subscript such that $a_j = 2k-1$, and b_j must be one of the $8-k$ numbers in $\{2k, 2k+2, ..., 14\}$; the remaining 6 even integers can be arranged in 6! ways.

The total of the payments by B to A for the $(7!)^2$ possible rounds is then

$$\sum_{k=1}^{7} 7!6!(8-k) = 7!6!(7+6+\cdots+1) = 7!6!28 = (7!)^2 4.$$

A pays to B a total of $(7!)^2 c$; so $c = 4$ makes the game fair. □

Solution II. For those who know that the sum of expected values (averages) is the expected value of the sum, we note that in the first spot, 1, 3, 5, 7, 9, 11, 13 are equally likely, and hence the average payment is the average of the payments for each of these numbers, 7/7, 6/7, 5/7, 4/7, 3/7, 2/7, 1/7. Hence in the first spot the average payment is

$$\frac{(7/7+6/7+5/7+4/7+3/7+2/7+1/7)}{7} = 4/7 \text{ of a cent.}$$

Since there are 7 spots, with an average payment of 4/7 cent each, the total payment averages 4 cents; so $c = 4$ makes the game fair. □

Problem 5. $\triangle ABC$ is inscribed in a circle k with center O so that $\angle ACB = 120^o$.

(a) If H is the orthocenter of $\triangle ABC$, prove that A, B, O, H lie on a circle with center the midpoint of the arc ACB. (The orthocenter of $\triangle ABC$ is the intersection point of its three altitudes.)

(b) If G is the centroid of $\triangle ABC$, and I is the incenter of $\triangle ABH$, prove that the points O, G, I, H lie on a line. (The centroid of $\triangle ABC$ is the intersection point of its three medians: a median connects a vertex of $\triangle ABC$ with the midpoint of the opposite side; the incenter of $\triangle ABC$ is the intersection of its three angle bisectors.)

Solution.[1] Let O_1 be the midpoint of the arc ACB and let R be the radius of k.

(a) Solution I. $\triangle AOO_1$ and $\triangle BOO_1$ are equilateral ($\angle ACB = 120^o \Rightarrow \angle AOB = 120^o$.) The segments AB and OO_1 intersect each other in their midpoint, D. If line AO intersects k in point P, then $\angle ABP = \angle ACP = 90^o$, i.e. $PB||CH$ and $PC||BH$, and thus $PBHC$ is a parallelogram. From here, $CH = PB$. Since OD is a midsegment in $\triangle ABP$, then $PB = 2OD = OO_1$ and $CH = OO_1 = R$. Again, $OCHO_1$ is a parallelogram; moreover, it is a rhombus for $OC = R$. Thus, $O_1H = R$ and the points A, B, O, H lie on a circle k_1 with center O_1 and radius R. □

(a) Solution II. Look at the quadrilateral $CMHN$: it contains two right angles (at M and N), and the angle at C is 120^o, so the angle at H is 60^o: $\angle AHB = 60^o$. But $\angle AOB = 120^o$ (as above), so $AHBO$ do lie on the same circle. Since A, O, B lie on a circle with center O_1 (as above), H is forced to lie on the same circle. □

[1] This problem was given at a national contest in Bulgaria in 1994.

(b) Solution I. In k_1, $\angle BO_1H$ is central, and $\angle BAH$ is inscribed, so that $\angle BO_1H = 2\angle BAH$ and $\angle BHO_1 = 90^o - \angle BAH = \angle AHC$. In the rhombus $OCHO_1$ the diagonal OH is the bisector of $\angle CHO_1$. Hence, $\angle AHO = \angle AHC + \angle CHO = \angle BHO_1 + \angle OHO_1 = \angle BHO \Rightarrow OH$ is the angle bisector also of $\angle AHB$. This means that point I lies on OH.

Let OH intersect the median CD in point G. We will show that G is the medicenter of $\triangle ABC$ by showing first that $CG = 2GD$. Indeed, if E and F are the midpoints of CG and HG, then EF is a midsegment in $\triangle CHG$ with $EF = CH/2 = OD$ and $EF || OD$ (from (a)). Then $\triangle EFG \cong \triangle DOG$, and $EG = GD$, $CG = 2EG = 2GD$. Thus, G is the medicenter of $\triangle ABC$, and the points I and G lie on the line OH. $\square$

(b) Solution II. For those who know about the *Euler line*: points O, G, H lie on the Euler line of $\triangle ABC$ (we even know the ratio $OG : GH = 1 : 2$.) Thus, it remains to show that I lies on the line OH, or equivalently, that OH is the angle bisector of $\angle AHB$. Recall that k_1 was the circle described around $AHBO$ from part (a). Since $OA = OB$, the corresponding arcs OA and OB on k_1 are equal, and hence the inscribed angles are equal: $\angle AHO = \angle BHO$. $\square$