BAMO 1999: PRACTICE PROBLEMS

BAMO PROBLEMS COMMITTEE

Note: You have 4 hours to solve as many problems as you can from the following list of 5 problems. Each solution should be written clearly and in detail on a separate sheet of paper. Each problem is worth 10 points. Partial credit will be awarded for partial solutions.

Problem 1. Let A and B be two different hospitals that treat exactly the same number of patients during a year. Each patient suffers from one of two diseases, X or Y. Hospital A cures a greater percentage of its patients than hospital B. Is it possible that hospital B cures both a greater percentage of X–patients than A, and a greater percentage of Y–patients than A?

Problem 2. Bildert works in a cubicle in an office which consists of 27 cubicles arranged in a $3 \times 3 \times 3$ cube. Any two cubicles sharing a wall have a connecting door on this wall; for example, the corner cubicles have exactly 3 doors, while the center cubicle has 6 doors: one on each wall, one on the floor, and one on the ceiling. If Bildert starts at the central cubicle, can he visit each of the other 26 cubicles exactly once (i.e. without revisiting any cubicles)?

Problem 3. Let a, b, c, d, e, f be positive integers, each at least 2, whose sum is S. Prove that

$$a(a-1)+b(b-1)+c(c-1)+d(d-1)+e(e-1)+f(f-1) \leq (S-10)(S-11)+10.$$

When is equality achieved?

Problem 4. In the $O-E$ game, a round starts with player A paying c cents to player B. Then A secretly arranges the numbers $1, 3, 5, 7, 9, 11, 13$ in some order as a sequence $a_1, a_2, ..., a_7$, and B secretly arranges $2, 4, 6, 8, 10, 12, 14$ as a sequence $b_1, b_2, ..., b_7$. Finally, the players show their sequences and B pays A one cent for each i in $X = \{1, 2, 3, 4, 5, 6, 7\}$ such that $a_i < b_i$. This finishes the round. What number c would make the game fair? (The game is fair if the total payments by A to B equals the total payments by B to A if all possible distinct rounds are played exactly once.)

Problem 5. $\triangle ABC$ is inscribed in a circle with center O so that $\angle ACB = 120^o$.

(a) If H is the orthocenter of $\triangle ABC$, prove that A, B, O, H lie on a circle with center the midpoint of the arc ACB. (The orthocenter of $\triangle ABC$ is the intersection point of its three altitudes.)

(b) If G is the centroid of $\triangle ABC$, and I is the incenter of $\triangle ABH$, prove that the points O, G, I, H lie on a line. (The centroid of $\triangle ABC$ is the intersection point of its three medians: a median connects a vertex of $\triangle ABC$ with the midpoint of the opposite side; the incenter of $\triangle ABC$ is the intersection of its three angle bisectors.)