BAMO 1999: PRACTICE PROBLEMS II

BAMO PROBLEMS COMMITTEE
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Participation in BAMO is already an achievement - four hours thinking over hard math problems is what counts, not who solved how many problems! The problems are arranged in approximately increasing difficulty; in particular, problems 4 and 5 are quite hard. We expect no more than 1 or 2 students (if any) to solve all problems, and few to solve completely 3 problems. Thus, solving even one problem is already a FINE performance! Remember that you are competing against students in your age group – if the problems seem hard (simple) to you, they are probably hard (simple) for the other contestants too. The idea of BAMO is that students find it a fun experience and give it all they have during the contest; and after the contest is over, students continue thinking over the proposed problems.

Problem 1. Prove that $1993^{1993} + 1994^{1994} + 1995^{1995} + 1996^{1996}$ is divisible by 10.

Problem 2. King Arthur and his 100 knights are having a feast at a round table. Each person has a glass with white or red wine in front of him. Exactly at midnight, every person moves his glass in front his left neighbor if the glass has white wine, or in front of his right neighbor if the glass has red wine. It is known that there is at least one glass with red and one glass with white wine.

(a) Prove that after midnight there will be at least one person without a glass of wine in front of him.

(b) If Lady Guinivera, the Queen, calls the King off the table before midnight, will the conclusion of part (a) still going to be correct?

Problem 3. On the bases AB and CD of a trapezoid $ABCD$ draw two squares externally to $ABCD$. Let O be the intersection point of the diagonals AC and BD, and let O_1 and O_2 be the centers of the two squares. Prove that O_1, O and O_2 lie on a line (i.e. they are *collinear*; see Figure on the back.)

Problem 4. The real positive numbers $a_1, a_2, ..., a_n, ...$ satisfy the relation $a_{n+1}^2 = a_n + 1$ for all $n = 1, 2, ...$ Prove that at least one of the a_i's must be an irrational number.

Problem 5. **[Simpson's Line]** Let $\triangle ABC$ be inscribed in a circle k, and let M be an arbitrary point on k different from A, B, C. Prove that the feet of the three perpendiculars from M to the sides of $\triangle ABC$ are collinear. (Note: You may have to extend some sides to find these feet: see Figure on the back.)

Problems 3 and 5