# Berkeley Math Circle Probability Jan. 31, 1999

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**Problem 1.** Two players alternately toss a penny, and the one that first tosses heads wins. What is the probability that

(a) the game never ends?
(b) the first player wins?
(c) the second player wins?

**Problem 2.** On average, how many times must a die be thrown until one gets a 6?

**Problem 3.** In a game of craps, a person tosses two dice which sum to 4. Then she tosses the two dice again and if the sum of 4 comes up, she wins; if the sum of 7 comes up, she loses; and if neither comes up she keeps tossing. What is the probability that she wins?

**Problem 4.** A box contains $a + b$ frogs, divided into two partitions, with $a$ frogs in one partition and $b$ frogs in the other partition. The wall between the partitions is low enough so that frogs can freely hop over it. Every minute, a frog randomly hops from one side to the other, with equal probability, until one side is empty, in which case the experiment ends. What is the probability that

(a) the experiment never ends?
(b) the partition starting with $a$ frogs ends up empty?
(c) the partition starting with $b$ frogs ends up empty?

**Problem 5.** Coupons in cereal boxes are numbered from 1 to 5, and a set of one of each is required for a prize. With one coupon per box, how many boxes on the average are required to make a complete set?

**Problem 6.** We randomly place $c$ cats and $d$ dogs in a row. On the average, how many pairs of adjacent positions are "heterospecious," i.e. dog-cat or cat-dog?

**Problem 7.** Given any sequence of $n$ distinct integers, we compute its "swap number" in the following way: Reading from left to right, whenever we reach a number which is less than the first number in the sequence, we swap its position with the first number in the sequence. We continue in this way until we get to the end of the sequence. The swap number of the sequence is the total number of swaps. For example, the sequence $3, 4, 2, 1$ has a swap number of 2, for we swap 3 with 2 to get $2, 4, 3, 1$ and then we swap 2 with 1 to get $1, 4, 3, 2$.

Find the average value of the swap number of all $n!$ permutations of the $n$ numbers.

**Problem 8.** Akira, Betül and Cleopatra fight a 3-cornered pistol duel. All three know that Akira's chance of hitting any target is 0.3, while Betül *never* misses, and Cleopatra has a 0.5 chance of hitting any target. The way the duel works is that each person is to fire at their choice of target, starting with Akira, and proceeding cyclically in alphabetical order (unless someone is hit, in which case they don't shoot), continuing until one person is left unhit. What is Akira's strategy?

**Problem 9.** Akira has $a$ dollars and Betül has $b$ dollars. They play a game in which the loser gives \$1 to the winner. Akira has a 2/3 chance of winning each time. They play until one is bankrupt. What is the probability that Akira wins? Generalize the above by replacing 2/3 with $p$, for $0 < p < 1$.

**Problem 10.** Shuffle an ordinary deck of 52 playing cards. On the average, how far from the top will the first ace be?

**Problem 11.** Place $n$ letters at random into $n$ envelopes. What is the average number of letters which get into the correct envelopes?

**Problem 12.** (USAMO 1973) Three distinct vertices are chosen at random from the vertices of a given regular $(2n+1)$-gon. If all such choices are equally likely, what is the probability that the center of the given polygon lies in the interior of the triangle determined by the three chosen random points?

**Problem 13.** (USAMO 1975) A deck of $n$ playing cards, which contains three aces, is shuffled at random (it is assumed that all possible card distributions are equally likely). The cards are then turned up one by one from the top until the second ace appears. Prove that the expected (average) number of cards to be turned up is $(n+1)/2$.

**Problem 14.** Devise a game using an ordinary fair coin in which the probability of winning is

(a) 1/3,

(b) any real number between 0 and 1.

**Problem 15.** You arrive in Las Vegas \$100 decide to play roulette, making the same bet each time, until you are either bankrupt or have doubled your money? Which strategy is best?

(a) Making bets of \$1 each time.

(b) Making bets of \$10 each time.

(c) Making a single bet of \$100.

**Problem 16.** Will a one-dimensional random walk return? How about 2-dimensional? 3-dimensional?