BERKELEY MATH CIRCLE 1998-99

Plane Geometry. Part I

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During the session, we discussed a way to find the length of curves by using polygonal approximation. (I encourage you to try this with an ellipse or the missing middle thirds set.) We then tried to find the area of a cylinder with polyhedral approximations. This turned out not to always work so well, which we might have guessed from "visual" geometric considerations. The fact remains, however, that noone really did guess it until Schwarz around the turn of the last century, and it is a worthwhile exercise to review carefully the calculation (limits, triangle inequality, trig formulas, etc.).

Problem 1. If you fold a paper without overlapping, and color the parts that get folded... which pieces of paper will have some places uncolored?

Hint: Let $p_1, ..., p_n$ be points in the plane. Also, let $m_1, ..., m_n$ be positive numbers. Given a line L in the plane let $d_1, ..., d_n$ be the distances from the points to L. Let $p_1, ..., p_k$ be the points on one side of L and $p_{k+1}, ..., p_n$ the points on the other side. If $d_1m_1 + ... + d_km_k$ is not equal to $d_{k+1}m_{k+1} + ... + d_nm_n$, then show that you can find a line parallel to L so that the corresponding sums will be equal.

Problem 2. Of all triangles which overlap the unit disk in an area 3, which one encloses the least perimeter of the circle.

Hint: Guess the answer first.

Problem 3. If a quadrilateral has side lengths a, b, c, d (in that order), the area it encloses does not exceed $(ac + bd)/2$.

Problem 4. Find the area of the triangle formed by the medians of a given triangle (in terms of the area of the given triangle).

Problem 5. Show that $\sin x$ is concave when $0 < x < \pi/2$. In other words, show that $\sin x > \sin y \cdot (x/y)$ when $0 < x < y < \pi/2$ (without calculus).