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Inversion in the Plane. Part II: Radical Axes[1]

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Definition 2. The *degree* of point A with respect to a circle $k(O, R)$ is defined as

$$d_k(A) = OA^2 - R^2.$$

This is simply the square of the tangent segment from A to k. Let M be the midpoint of AB in $\triangle ABC$, and CH – the altitude from C, with $H \in AB$ (cf. Fig.5-6.) Mark the sides BC, CA and AB by a, b and c, respectively. Then

$$|a^2 - b^2| = |BH^2 - AH^2| = c|BH - AH| = 2c \cdot MH, \tag{1}$$

where M is the midpoint of AB.

Definition 3. The *radical axis* of two circles k_1 and k_2 is the geometric place of all points which have the same degree with respect to k_1 and k_2: $\{A \,|\, d_{k_1}(A) = d_{k_2}(A)\}$.

Fig. 5-7

Let P be one of the points on the radical axis of $k_1(O_1, R_1)$ and $k_2(O_2, R_2)$ (cf. Fig.7.) We have by (1):

$$PO_1^2 - R_1^2 = PO_2^2 - R_2^2 \;\Rightarrow\; |R_1^2 - R_2^2| = |PO_1^2 - PO_2^2| = 2O_1O_2 \cdot MH,$$

where M is the midpoint of O_1O_2, and H is the orthogonal projection of P onto O_1O_2. Then

$$MH = \frac{|R_1^2 - R_2^2|}{2O_1O_2} = \text{constant} \;\Rightarrow\; \text{point } H \text{ is constant.}$$

(Show that the direction of $MH^{\rightarrow}$ is the same regardless of which point P on the radical axis we have chosen.) Thus, the radical axis is a subset of a line $\perp O_1O_2$. The converse is easy.

Lemma 1. *Let $k_1(O_1, R_1)$ and $k_2(O_2, R_2)$ be two nonconcentric circles circles, with $R_1 \geq R_2$, and let M be the midpoint of O_1O_2. Let H lie on the segment MO_2, so that*

$$HM = (R_1^2 - R_2^2)/2O_1O_2.$$

Then the radical axis of $k_1(O_1, R_1)$ and $k_2(O_2, R_2)$ is the line l, perpendicular to O_1O_2 and passing through H.

What happens with the radical axis when the circles are concentric? In some situations it is convenient to have the circles concentric. In the following fundamental lemma, we achieve this by applying both ideas of inversion and radical axis.

Lemma 2. *Let k_1 and k_2 be two nonintersecting circles. Prove that there exists an inversion sending the two circles into concentric ones.*

PROOF: If the radical axis intersects O_1O_2 in point H, let $k(H, \mathrm{d}_{k_i}(H))$ intersect O_1O_2 in A and B. Apply inversion wrt $k'(A, AB)$ (cf. Fig. 8.) Then $I(k)$ is a line l through B, $l \perp O_1O_2$. But $k_1 \perp k$, hence $I(k_1) \perp l$, i.e. the center of $I(k_1)$ lies on l. It also lies on O_1O_2, hence $I(k_1)$ is centered at B. Similarly, $I(k_2)$ is centered at B. □

FIG. 8

1. WARM-UP PROBLEMS

Problem 19. The radical axis of two intersecting circles passes through their points of intersection.

Problem 20. The radical axes of three circles intersect in one point, provided their centers do not lie on a line.

Problem 21. Given two circles k_1 and k_2, find the geometric place the centers of the circles k perpendicular to both k_1 and k_2.

2. Problems

Problem 22. A circle k is tangent to a line l at a point P. Let O be diametrically opposite to P on k. For some points $T, S \in k$ set $OT \cap l = T_1$ and $OS \cap l = S_1$. Finally, let SQ and TQ be two tangents to k meeting in point Q. Set $OQ \cap l = \{Q_1\}$. Prove that Q_1 is the midpoint of T_1S_1.

Problem 23. Consider $\triangle ABC$ and its circumscribed and inscribed circles K and k, respectively. Take an arbitrary point A_1 on K, draw through A_1 a tangent line to k and let it intersect K in point B_1. Now draw through B_1 another tangent line to k and let it intersect K in point C_1. Finally, draw through C_1 a third tangent line to k and let it intersect K in point D_1 (cf. Fig. 9.) Prove that D_1 coincides with A_1. In other words, prove that any triangle $A_1B_1C_1$ inscribed in K, two of whose sides are tangent to k, must have its third side also tangent to k so that k is the inscribed circle for $\triangle A_1B_1C_1$ too.

Figures 9-11

Problem 24. Find the distance between the center P of the inscribed circle and the center O of the circumscribed circle of $\triangle ABC$ in terms of the two radii r and R.

Problem 25. We are given $\triangle ABC$ and points $D \in AC$ and $E \in BC$ such that $DE||AB$. A circle k_1 of diameter DB intersects a circle k_2 of diameter AE in M and N. Prove that M and N lie on the altitude CH to AB.

Problem 26. Prove that the altitude of $\triangle ABC$ through C is the radical axis of the circles with diameters the medians AM and BN of $\triangle ABC$.

Problem 27. Find the geometric place of points O which are centers of circles through the end points of diameters of two fixed circles k_1 and k_2.

Problem 28. Construct all radical axes of the four incircles of $\triangle ABC$.

Problem 29. Let A, B, C be three collinear points with B inside AC. On one side of AC we draw three semicircles k_1, k_2 and k_3 with diameters AC, AB and BC, respectively. Let BE be the interior tangent between k_2 and k_3 ($E \in k_1$), and let UV be the exterior tangent to k_2 and k_3 ($U \in k_2$ and $V \in k_3$). Find the ratio of the areas of $\triangle UVE$ and $\triangle ACE$ in terms of k_2 and k_3's radii.(cf. Fig. 10)

Problem 30. The chord AB separates a circle γ into two parts. Circle γ_1 of radius r_1 is inscribed in one of the parts and it touches AB at its midpoint C. Circle γ_2 of radius r_2 is also inscribed in the same part of γ so that it touches AB, γ_1 and γ. Let PQ be the interior tangent of γ_1 and γ_2, with $P, Q \in \gamma$. Show that $PQ \cdot SE = SP \cdot SQ$, where $S = \gamma_1 \cap \gamma_2$ and $E = AB \cap PQ$.(cf. Fig. 11)

Problem 31. Let $k_1(O, R)$ be the circumscribed circle around $\triangle ABC$, and let $k_2(T, r)$ be the inscribed circle in $\triangle ABC$. Let $k_3(T, r_1)$ be a circle such that there exists a quadrilateral $AB_1C_1D_1$ inscribed in k_1 and circumscribed around k_3. Calculate r_1 in terms of R and r.

Problem 32. Let $ABCD$ be a square, and let l be a line such that the reflection A_1 of A across l lie on the segment BC. Let D_1 be the reflection of D across l, and let D_1A_1 intersect DC in point P. Finally, let k_1 be the circle of radius r_1 inscribed in $\triangle A_1CP_1$. Prove that $r_1 = D_1P_1$.

Problem 33. In a circle $k(O, R)$ let AB be a chord, and let k_1 be a circle touching internally k at point K so that $KO \perp AB$. Let a circle k_2 move in the region defined by AB and not containing k_1 so that it touches both AB and k. Prove that the tangent distance between k_1 and k_2 is constant.

Problem 34. Prove that for any two circles there exists an inversion which transforms them into congruent circles (of the same radii). Prove further that for any three circles there exists an inversion which transforms them into circles with collinear centers.

Problem 35. Given two nonintersecting circles k_1 and k_2, show that all circles orthogonal to both of them pass through two fixed points and are tangent pairwise.

Problem 36. Given two circles k_1 and k_2 intersecting at points A and B, show that there exist exactly two points in the plane through which there passes no circle orthogonal to k_1 and k_2.

3. Problems From Around the World

Problem 37 (IMO Proposal)**.** The incircle of $\triangle ABC$ touches BC, CA, AB at D, E, F, respectively. X is a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches BC at D also, touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

Problem 38 (Israel, 1995)**.** Let PQ be the diameter of semicircle H. Circle k is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that AB is perpendicular to PQ and is also tangent to k. Prove that AC bisects $\angle PAB$.

Problem 39 (Romania, 1997)**.** Let ABC be a triangle, D a point on side BC, and ω the circumcicle of ABC. Show that the circles tangent to ω, AD, BD and to ω, AD, DC are also tangent to each other if and only if $\angle BAD = \angle CAD$.

Problem 40 (Russia, 1995)**.** We are given a semicircle with diameter AB and center O, and a line which intersects the semicircle at C and D and line AB at M ($MB < MA$, $MD < MC$.) Let K be the second point of intersection of the circumcircles of $\triangle AOC$ and $\triangle DOB$. Prove that $\angle MKO = 90°$.

4. Final Remarks on Inversion: Alternative Definition of Inversion in Terms of Complex Numbers

The points in the usual coordinate plane P can be thought of as complex numbers: the point $A = (a, b)$ can be thought of as the complex number $z = a + bi$ with $a, b \in \mathbb{R}$. Thus, the x-coordinate of A corresponds to the *real part* of z: $\mathbb{R}\mathrm{e}(z) = a$, and the y-coordinate of A corresponds to the *imaginary part* of z: $\mathbb{I}\mathrm{m}(z) = b$. Recall how we add and subtract complex numbers: this corresponds exactly to addition and subtraction of vectors originating at (0,0) in the plane. For instance, if $z_1 = a_1 + b_1 i$, then $z + z_1 = (a + a_1) + (b + b_1)i$; this corresponds exactly to what would happen if we add two vectors $\vec{v}$ and $\vec{v}_1$ which start at the origin and end in (a, b) and (a_1, b_1), respectively: $\vec{v} + \vec{v}_1$ would start at the origin and end in $(a + a_1, b + b_1)$ (cf. Fig. 12.)

Multiplication of complex numbers can be also translated in terms of vectors in the plane. To multiply z and z_1 from above, we perform the usual algebraic manipulations:

$$z \cdot z_1 = (a + bi) \cdot (a_1 + b_1 i) = aa_1 + ab_1 i + ba_1 i + bb_1(i^2) = (aa_1 - bb_1) + (ab_1 + ba_1)i.$$

FIGURES 12-13

The resulting "vector" $\vec{v}'$ from this multiplication corresponds to $(aa_1 - bb_1, ab_1 + ba_1)$, and it can be interpreted geometrically from the starting vectors $\vec{v}$ and $\vec{v}_1$. I urge you to check in a few simple examples that $\vec{v}'$ can be described as follows: add the angles that $\vec{v}$ and $\vec{v}_1$ form with the x-axis – this is going to be direction of $\vec{v}'$; for the length of $\vec{v}'$, take the product of the lengths of $\vec{v}$ and $\vec{v}_1$. (Hint: use the so-called "polar form" of vectors and some simple trigonometric identities.)

Question 1. What does this have to do with Inversion?

The function Inversion from the plane P to P, as we defined it earlier, can be viewed simply as a complex function, i.e. a function whose input and output are complex numbers. To explain this, we need to introduce one further notion: the *conjugate* of a complex number. If $z = a + bi$ is a complex number, then the conjugate of z, denoted by $\overline{z}$, is simply the complex number obtained from be z by switching the sign of z's imaginary part: $\overline{z} = a - bi$. Geometrically, the points (a, b) and $(a, -b)$ are reflections of each other across the x-axis (cf. Fig. 13.) The "miraculous" property of conjugates is that their product is always a *real* number:

$$z \cdot \overline{z} = (a + bi) \cdot (a - bi) = a^2 + b^2 \in \mathbb{R}.$$

Now we are ready to define Inversion in terms of complex numbers:

Lemma 3. *The function Inversion $I : P \to P$, with center $O = (0, 0)$ and radius $r = 1$, can be described alternatively by identifying the coordinate plane P with the plane of complex numbers $\mathbb{C}$, and defining the image of $A = (a, b)$ to be the complex number:*

$$I(A) = \frac{1}{\overline{z}},$$

where $z = a + bi \in \mathbb{C}$ is the complex number corresponding to A.

In other words, Inversion sends the "point" $z = a + ib$ to the "point" $\dfrac{1}{\overline{z}}$. The latter has some coordinates produced by the division of the numbers 1 and $\overline{z}$. Of course, you can say – but how can we divide two complex numbers and get a third complex complex number? Here is an example of how this is done:

$$\frac{1 - 3i}{2 + 7i} = \frac{(1 - 3i)(2 - 7i)}{(2 + 7i)(2 - 7i)} = \frac{-19 - 15i}{4 + 49} = -\frac{19}{53} - \frac{15}{53}i.$$

Here we multiplied the numerator and denominator of the original fraction by $(2-7i)$, (the conjugate of $2 + 7i$), which forced the denominator to become a real number (53), and as a result we ended up with an "ordinary" complex number.

Thus, according to the lemma, to find where Inversion sends the point $A = (1, 1)$, we consider the complex number $z = 1 + i$, and find the corresponding complex number $1/\overline{z}$:

$$\frac{1}{\overline{z}} = \frac{1}{1 - i} = \frac{1 + i}{(1 - i)(1 + i)} = \frac{1 + i}{2} = \frac{1}{2} + \frac{1}{2}i.$$

Thus, $A = (1,1)$ will be sent by the Inversion to the point $A_1 = (\frac{1}{2}, \frac{1}{2})$. Well, it is easy to check that A_1 will be indeed the image of A under Inversion: note that A_1 lies on the segment OA, and $|OA| \cdot |OA_1| = \sqrt{2}\sqrt{1/2} = 1$. We urge the reader to *prove* the above lemma by using the elementary properties of complex numbers above and the original definition of Inversion.

Question 2. How good is this new interpretation of Inversion? The original definition seems quite alright, and besides, it does not require knowing complex numbers at all?!

Consider how many cases we have to go through in order to see what happens to circles and lines under Inversion: 4 cases. In addition, the *proof* of "preservation of angles" under Inversion requires us to look at all possible pairs of cases above, making it quite an unattractive work to sweat over ... 10 cases! Besides, the proof in each case has little or no relevance to the other cases, that is, we cannot find one general explanation for *why angles should be preserved under Inversion*! And honestly speaking, going through all proofs in 10 cases does not really "impart on us more wisdom": it only produces technical explanations; we have now no better idea of why Inversion has its wonderful properties than before we started!

In search of a better unifying explanation of why Inversion can do all the miraculous things it does, we invoke the theory of complex functions.

Thus, we consider *complex functions* $f : \mathbb{C} \to \mathbb{C}$, that is, functions with complex numbers as input and output. For example, $f(z) = z$, $f(z) = 3z^2$, $f(z) = \overline{z}$, $f(a+ib) = a+2abi$ are all complex functions. We can also look at functions f defined not on the whole complex plane $\mathbb{C}$, but just on some nice subset of it. For example, $f(z) = 1/z$ for $z \neq 0$, and $f(z) = 1/\overline{z}$, for $z \neq 0$.

As with *real functions* (e.g. $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2 - 4x$,) we can define *differentiability* of complex functions. We say that a function $f : U \to \mathbb{C}$, where U is an open subset of $\mathbb{C}$, is *complex differentiable* at $z_0 \in U$ if the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. We denote this limit, as usual, by $f'(z)$. In order not to confuse this definition with the *real differentiability*, we call a complex differentiable function f *holomorphic*.

So far so good, except that it is not so obvious when a complex function is holomorphic. We can though describe a whole class of obviously holomorphic functions: these will be polynomials and rational functions of z, e.g. $f(z) = z$, $f(z) = z+3z^2$, $f(z) = 1/z$, but *not* $f(z) = f(a+bi) = a+2abi$. I shall not elaborate here more on the subject, but just point out a good reference: *Complex Analysis*, by Serge Lang, Springer-Verlag.

In any case, the story goes roughly as follows.

Theorem 1. *Any holomorphic function preserves angles.*

More precisely, given two paths in the plane (cf. Fig. 14) meeting at point $A = (a, b)$, we assume that the tangent lines t_1 and t_2 at A to both paths exist. Set α to be the angle between t_1 and t_2. After applying a holomorphic function f, we transform the two paths into some other paths $f(path\,1)$ and $f(path\,2)$, and they meet at point $B = f(z_0)$. Set α to be the angle between t_1 and t_2. After applying a holomorphic function f, we transform the two paths into some other paths $f(path\,1)$ and $f(path\,2)$, and they meet at point $B = f(z_0)$. Then, the theorem asserts that the new paths will also have tangent lines at B, which will make precisely the same angle α with each other. In other words, the angle between the original paths is preserved.

Now, Inversion is not quite a holomorphic function (if it were $f(z) = 1/z$ it would have been holomophic everywhere except for $z = 0$, where it is not defined anyway.) But inversion $f(z) = 1/\overline{z}$

belongs to a class of functions, called, "antiholomorphic": roughly speaking, these are functions "holomorphic" in the variable $\overline{z}$, *not* in z. Such functions *reverse* the angles between paths.[2] As far as the measure of the angles is concerned, it is always preserved under both holomorphic and antiholomorphic functions.

Thus, if the truth, only the truth and the whole truth is to be told,

Theorem 2. *Inversion in the plane reverses the angles between any two figures (paths) (as long as we can define such angles.)*

Problem 41. Use the formula $f(z) = r^2/\overline{z}$ to describe directly the images of circles and lines passing through (or not through) the center of inversion.

5. Hints and Solutions to Selected Problems in Part I and Part II

Note: If a radius of an inversion is not specified, then assume that it is arbitrary.

Hint 1-2. $I(A, r)$.

Hint 3. If $r_1 \leq r_2 \leq r_3$, reduce to #1 or #2 by replacing k_1 by its center, k_2 by $k_2'(O_2, r_2 - r_1)$, and k_3 by $k_3'(O_2, r_3 - r_1)$.

Hint 4. If K, S, Q are the centers if k, s, q, respectively, then the wanted geometric place is the incircle of $\triangle KSQ$: use $I(A, r)$.

Hint 5. Ist way: $I(A, r)$; IInd way: if k_2 and k_4 do not intersect, invert them into concentric circles (cf. Fig. 15-17); IIIrd way: forget about inversion, and notice that $O_1O_2O_3O_4$ is circumscribed around a circle.

Hint 6. $I(A_1, r)$.

Solution 7. Using $I(D, r)$, we have A_1, B_1, C_1 collinear and $A_1B_1 + B_1C_1 = A_1C_1$. Then

$$\frac{AB \cdot r^2}{DA \cdot DB} + \frac{BC \cdot r^2}{DB \cdot DC} = \frac{AC \cdot r^2}{DA \cdot DC} \;\Rightarrow\; AB \cdot DC + AD \cdot BC = AC \cdot BD. \quad \square$$

Hint 8. $I(P, r)$.

[2] Another way to see why Inversion *reverses* angles is to view Inversion as the composition of two functions: $f_1(z) = 1/z$ for $z \neq 0$, and the reflection along the x-axis, $f_2(z) = \overline{z}$: thus, $I(z) = f_2 \circ f_1$. Since f_1 preserves angles (it is holomorphic), and f_2 reverses angles (simple geometric verification), it follows that their composition I will reverse angles.

FIG. 15-17

Hint 9. If $k_1 \cap k_2 = \emptyset$, let $I(O, r)$ send them into concentric circles. Consider the cases when $I(k_3)$ is a circle, and when it is a line.

Hint 10-11. $I(P, r)$.

Hint 12. Describe circles k and k_1 around the squares, and use inscribed angles to show that the point in question is the intersection point of k and k_1, other than D.

Hint 13. $I(D, r)$, then $A_1B_1 = C_1B_1$.

Hint 14. $I(D, r)$, then $\angle A_1B_1C_1 = \beta$ in $\triangle A_1B_1C_1$.

Solution 15. Apply $I(A, r)$ (cf. Fig.18-19.) Then $AD_1B_1C_1$ is a parallelogram, and $E_1 \in AB_1^{\rightarrow}$. But $AE_1 \cdot AE = r^2 = AB_1 \cdot AB$, and $AE = 2AB$. Hence $AE_1 = AB_1/2$, and E_1 is the midpoint of AB_1, i.e. $AB_1 \cap D_1C_1 = \{E\}$. Since D_1, E_1, C_1 are collinear, then D, E, C, A are concyclic. $\square$

FIG. 18-19

Solution 16. Apply $I(O, r)$, and let $\triangle ABC \xrightarrow{I} \triangle A_1B_1C_1$ (cf. Fig.20.) If $\angle AOB = \gamma$, $\angle OA_1B_1 = \alpha$, and $\angle OB_1A_1 = \beta$, then

$$\frac{OA + OB}{OC} = \frac{OC_1}{OA_1} + \frac{OC_1}{OB_1} = \frac{\sin\alpha}{\sin(\alpha + \gamma/2)} + \frac{\sin\beta}{\sin(\alpha + \gamma/2)} = 2\cos\frac{\gamma}{2}. \quad \square$$

Hint 17. Let $I(O, r)$ send k and k^* into concentric circles.

Solution 22. Apply $I(O, OP)$ (cf. Fig.21.) Then $I(k) = l$, so that $I(T) = T_1$ and $I(S) = S_1$ are the given points on l. Let $k_q(Q, QS = QT)$, and let X be the center of $I(k_q)$. We know that $X \in OQ^{\rightarrow}$. But since $k_q \perp k$, then $I(k_q) \perp l$, i.e. l passes through the center X. Thus, $X = Q_1$, and $XT_1 = XS_1$ implies that Q_1 is the midpoint of T_1S_1. $\square$

FIG. 20-21

Solution 18. We refer to Fig.22-23 for notation. Let $k_3 \cap l = \{B\}$, and let the radius R be chosen so that the circle $k_0(B, R) \perp k$. Apply $I(B, R)$. Then $I(k) = k(O, 1)$, $I(l) = l$, $I(k_3) = k_3'$ is a line parallel to l, and $I(k_1) = k_1'(O', R')$ and $I(k_2) = k_2'$ are both circles, touching l and k_3', and hence, of the same radii R'. If A' is their point of tangency, from $\triangle O'AO$: $(R')^2 + ((R')^2 - 1)^2 = ((R')^2 + 1)^2 \Rightarrow R' = 4$. The wanted distance is $2R' - 1 = 7$. $\square$

FIG. 22-23

Solution 24. Let A_0, B_0, C_0 be the points of tangency of the inscribed circle $k_P(P, r)$ with BC, CA, AB, respectively (cf. Fig.24.) Let A_1, B_1, C_1 be the midpoints of the corresponding sides of $\triangle A_0B_0C_0$. Note that $A_1 = PA \cap C_0B_0$, and similarly for B_1 and C_1. Apply $I(P, r)$. Since $PA \perp C_0B_0$, it follows $I(A) = A_1$, and similarly, $I(B) = B_1$, $I(C) = C_1$. Thus, the circumscribed circle $k(O, R)$ goes under I to the circumscribed circle around $\triangle A_1B_1C_1$, $k_1(O_1, r_1)$. Note that k_1 is half the size of k_P, hence $r_1 = r/2$.

FIG. 24-25

This is enough to find the distance between O and P. More generally, the distance d between the center O of a circle and the center P of an inversion satisfies:

$$r_1 \cdot |d^2 - R^2| = r^2 \cdot R, \tag{2}$$

where r is the radius of the inversion, R is the original radius of the circle, and r_1 is the radius of the image circle. Indeed, (cf. Fig.25,) for the diametrically opposite points $S, T \in PO \cap k$, and for their images $S_1, T_1 \in PO \cap k_1$ we have:

$$S_1T_1 = \frac{ST \cdot r^2}{PS \cdot PT} \;\Rightarrow\; 2r_1 = \frac{2R \cdot r^2}{|d-R|(d+R)},$$

from where (2) follows. Substituting $r_1 = r/2$, and the obvious relation $R > d$, we get $d = \sqrt{R(R-2r)}$. Note that for $d = 0$ we have an equilateral $\triangle ABC$, and $R = 2r$. □

Solution 25. Let A_1 and B_1 be the orthogonal projections of A and B onto BC and AC, and let k_3 be the circle through A, B, A_1, B_1 (cf. Fig. 26.) The radical axis of k_1 and k_2 is MN, the radical axis of k_1 and k_3 is AA_1, and the radical axis of k_2 and k_3 is BB_1. Hence they all intersect in one point H, the orthocenter of $\triangle ABC$. Thus, $H \in CC_1 \cap MN$.

Let O_1 and O_2 be the centers of k_1 and k_2, or equivalently, the midpoints of BD and AE. Since $DE||AB$, then $O_1O_2||AB$. But $MN \perp O_1O_2$, so $MN \perp AB$, i.e. $MN||CC_1$. Since the last two intersect in point H, it follows that they coincide as lines. □

Fig. 26-27

Hint 26. Set $D \equiv E \equiv C$ in #25.

Hint 27. Consider the radical axis of the two circles and its reflection across the perpendicular bisector of O_1O_2 (cf. Fig.27.)

Hint 28. Consider the intersection of a radical axis of two of the circles with a tangent to these two circles.

Hint 29. $I(E, r = EB)$ and show that U lies on line AE, and V lies on line EC. ALternatively, find a non-inversive proof that $BVEU$ is a rectangle.

Hint 30. $I(S, r)$.