Inversion in the Plane. Part I
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Note: All objects lie in the plane, unless otherwise specified. The expression “object $A$ touches object $B$” refers to tangent objects, e.g. lines and circles.

1. Definition of Inversion in the Plane

**Definition 1.** Let $k(O,r)$ be a circle with center $O$ and radius $r$. Consider a function on the plane, $I : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, sending a point $X \neq O$ to the point on the half line $OX$, $X_1$, defined by

$$OX \cdot OX_1 = r^2.$$ 

Such a function $I$ is called an *inversion of the plane* with center $O$ and radius $r$ (write $I(O,r)$.)

**Figures 1-2.**

It is immediate that $I$ is *not* defined at $p,O$. But if we compactify $\mathbb{R}^2$ to a sphere by adding one extra point $O_\infty$, we could define $I(O) = O_\infty$ and $I(O_\infty) = O$.

An inversion of the plane can be equivalently described as follows (cf. Fig.1.) If $X \in k$, then $I(X) = X$. If $X$ lies outside $k$, draw a tangent from $X$ to $k$ and let $X_2$ be the point of tangency. Drop a perpendicular $X_2X_1$ towards the segment $OX$ with $X_1 \in OX$, and set $I(X) = X_1$. The case when $X$ is inside $k$, $X \neq O$, is treated in a reverse manner: erect a perpendicular $XX_2$ to $OX$, with $X_2 \in k$, draw the tangent to $k$ at point $X_2$ and let $X_1$ be the intersection of this tangent with the line $OX$; we set $I(X) = X_1$. 
2. Properties of Inversion

Some of the basic properties of a plane inversion $I(O, r)$ are summarized below:

- $I^2$ is the identity on the plane.
- If $A \neq B$, and $I(A) = A_1, I(B) = B_1$, then $\triangle OAB \sim \triangle OB_1A_1$ (cf. Fig. 2.) Consequently,
  \[ A_1B_1 = \frac{AB \cdot r^2}{OA \cdot OB}. \]
- If $l$ is a line with $O \in l$, then $I(l) = l$.
- If $l$ is a line with $O \notin l$, then $I(l)$ is a circle $k_1$ with diameter $OM_1$, where $M_1 = I(M)$ for the orthogonal projection $M$ of $O$ onto $l$ (cf. Fig.3.)

Consequently,
  \[ A_1B_1 = \frac{AB \cdot r^2}{OA \cdot OB}. \]

3. Problems

Problem 1. Given a point $A$ and two circles $k_1$ and $k_2$, construct a third circle $k_3$ so that $k_3$ passes through $A$ and is tangent to $k_1$ and $k_2$. (cf. Fig.5)

Problem 2. Given two points $A$ and $B$ and a circle $k_1$, construct another circle $k_2$ so that $k_2$ passes through $A$ and is tangent to $k_1$. (cf. Fig.6)

Problem 3. Given circles $k_1, k_2$ and $k_3$, construct another circle $k$ which tangent to all three of them.
Problem 4. Let $k$ be a circle, and let $A$ and $B$ be points on $k$. Let $s$ and $q$ be any two circles tangent to $k$ at $A$ and $B$, respectively, and tangent to each other at $M$. Find the set traversed by the point $M$ as $s$ and $q$ move in the plane and still satisfy the above conditions. (cf. Fig.7)

Problem 5. Circles $k_1, k_2, k_3$ and $k_4$ are positioned in such a way that $k_1$ is tangent to $k_2$ at point $A$, $k_2$ is tangent to $k_3$ at point $B$, $k_3$ is tangent to $k_4$ at point $C$, and $k_4$ is tangent to $k_1$ at point $D$. Show that $A, B, C$ and $D$ are either collinear or concyclic. (cf. Fig.8)

Problem 6. Circles $k_1, k_2, k_3$ and $k_4$ intersect cyclicly pairwise in points $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{C_1, C_2\}$, and $\{D_1, D_2\}$. ($k_1$ and $k_2$ intersect in $A_1$ and $A_2$, $k_2$ and $k_3$ intersect in $B_1$ and $B_2$, etc.) (cf. Fig.9)

- Prove that if $A_1, B_1, C_1, D_1$ are collinear (concyclic), then $A_2, B_2, C_2, D_2$ are also collinear (concyclic).
- Prove that if $A_1, A_2, C_1, C_2$ are concyclic, then $B_1, B_2, D_1, D_2$ are also concyclic.

Problem 7. [Ptolemy’s Theorem] Let $ABCD$ be inscribed in a circle $k$. (cf. Fig.10) Prove that the sum of the products of the opposite sides equals the product of the diagonals of $ABCD$:

$$AB \cdot DC + AD \cdot BC = AC \cdot BD.$$ 

Further, prove that for any four points $A, B, C, D$:

$$AB \cdot DC + AD \cdot BC \geq AC \cdot BD.$$ 

When is equality achieved?

Problem 8. Let $k_1$ and $k_2$ be two circles, and let $P$ be a point. Construct a circle $k_0$ through $P$ so that $\angle(k_1, k_0) = \alpha$ and $\angle(k_1, k_0) = \beta$ for some given angles $\alpha, \beta \in [0, \pi)$. 

Figures 8-10.
Problem 9. Given three angles $\alpha_1, \alpha_2, \alpha_3 \in [0, \pi)$ and three circles $k_1, k_2, k_3$, two of which do not intersect, construct a fourth circle $k$ so that $\angle(k, k_i) = \alpha_i$ for $i = 1, 2, 3$.

Problem 10. Construct a circle $k^*$ so that it goes through a given point $P$, touches a given line $l$, and intersects a given circle $k$ at a right angle.

Problem 11. Construct a circle $k$ which goes through a point $P$, and intersects given circles $k_1$ and $k_2$ at angles $45^\circ$ and $60^\circ$, respectively.

Problem 12. Let $ABCD$ and $A_1B_1C_1D_1$ be two squares oriented in the same direction. Prove that $AA_1$, $BB_1$ and $CC_1$ are concurrent if $D \equiv D_1$.

Problem 13. Let $ABCD$ be a quadrilateral, and let $k_1, k_2$, and $k_3$ be the circles circumscribed around $\triangle DAC$, $\triangle DCB$, and $\triangle DBA$, respectively. Prove that if $AB \cdot CD = AD \cdot BC$, then $k_2$ and $k_3$ intersect $k_1$ at the same angle.

Problem 14. In the quadrilateral $ABCD$, set $\angle A + \angle C = \beta$.

- If $\beta = 90^\circ$, prove that that $(AB \cdot CD)^2 + (BC \cdot AD)^2 = (AC \cdot BD)^2$.
- If $\beta = 60^\circ$, prove that $(AB \cdot CD)^2 + (BC \cdot AD)^2 = (AC \cdot BD)^2 + AB \cdot BC \cdot CD \cdot DA$.

Problem 15. Let $k_1$ and $k_2$ be two circles intersecting at $A$ and $B$. Let $t_1$ and $t_2$ be the tangents to $k_1$ and $k_2$ at point $A$, and let $t_1 \cap k_2 = \{A, C\}$, $t_2 \cap k_1 = \{A, D\}$. If $E \in AB^\perp$ such that $AE = 2AB$, prove that $ACED$ is concyclic. (cf. Fig.11)

Problem 16. Let $OL$ be the inner bisector of $\angle POQ$. A circle $k$ passes through $O$ and $k \cap OP^\perp = \{A\}$, $k \cap OQ^\perp = \{B\}$, $k \cap OL^\perp = \{C\}$. (cf. Fig.12) Prove that, as $k$ changes, the following ratio remains constant:

$$\frac{OA + OB}{OC}$$

Problem 17. Let a circle $k^*$ be inside a circle $k$, $k^* \cap k = \emptyset$. We know that there exists a sequence of circles $k_0, k_1, ..., k_n$ such that $k_i$ touches $k, k^*$ and $k_{i-1}$ for $i = 1, 2, ..., n + 1$ (here $k_{n+1} = k_0$.) Show that, instead of $k_1$, one can start with any circle $k'_1$ tangent to both $k$ and $k^*$, and still be able to fit a “ring” of $n$ circles as above. What is $n$ in terms of the radii of and the distance between the centers of $k$ and $k^*$? (cf. Fig. 13)

Problem 18. Circles $k_1, k_2, k_3$ touch pairwise, and all touch a line $l$. A fourth circle $k$ touches $k_1, k_2, k_3$, so that $k \cap l = \emptyset$. Find the distance from the center of $k$ to $l$ given that radius of $k$ is 1. (cf. Fig. 14)