Problems for Approximately Rational Numbers by Dmitry Fuchs

1. Let $q = 2^m 5^n r$, $a = \max(m, n)$ and $b$ the minimal number of 9's such that $\underbrace{999...9}_b$ is divisible by $r$. Then in decimal representation, the fraction $\frac{p}{q}$ has period of length $b$, and $a$ initial decimal digits before its period starts.

2. (a) For every irrational $\alpha$ there exists infinitely many $\frac{p_k}{q_k}$ such that

$$|\alpha - \frac{p_k}{q_k}| < \frac{1}{\sqrt{5} q_k^2}.$$ 

(b) There are irrational numbers, $\alpha$ such that for every $\lambda > \sqrt{5}$ there exists finitely many $\frac{p}{q}$'s such that

$$|\alpha - \frac{p}{q}| < \frac{1}{\lambda q^2}.$$ 

(c) If $\alpha \neq$ the continued fraction $[a_0, a_1, a_2,...]$ where all but finitely many of the partial quotients, $a_k$'s, are not 1's then the $\sqrt{5}$ in part (a) can be replaced with $\sqrt{8}$

3. Given two vectors, $\mathbf{a}$ and $\mathbf{b}$, from the origin, let $a_0$ be the largest integral multiple of $\mathbf{b}$ that can be "added" to the tip of $\mathbf{a}$ without crossing the y-axis. Let $\mathbf{v_1}$ be the vector $\mathbf{a} + a_0 \mathbf{b}$. $a_1$ is the largest integral multiple of $\mathbf{v_1}$ that can be "added" to $\mathbf{b}$ without crossing the y-axis. Let $\mathbf{v_2}$ be the vector $\mathbf{b} + a_1 \mathbf{a}$. $a_2$ is the largest integral multiple of $\mathbf{v_1}$ that can be "added" to $\mathbf{a}$ without crossing the y-axis. Continuing in this way the sequence $a_0, a_1, a_2,...$ is formed. Show that $\frac{||\mathbf{a}||}{||\mathbf{b}||}$ equals the continued fraction $[a_0, a_1, a_2,...]$