LINEAR RECURSIVE SEQUENCES
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1. Sequences

A sequence is an infinite list of numbers, like
\[ 1, 2, 4, 8, 16, 32, \ldots. \] (1)
The numbers in the sequence are called its terms. The general form of a sequence is
\[ a_1, a_2, a_3, \ldots \]
where \( a_n \) is the \( n \)-th term of the sequence. In the example (1) above, \( a_1 = 1, a_2 = 2, a_3 = 4 \), and so on.

The notations \( \{a_n\} \) or \( \{a_n\}_{n=1}^{\infty} \) are abbreviations for
\[ a_1, a_2, a_3, \ldots. \]
Occasionally the indexing of the terms will start with something other than 1. For example, \( \{a_n\}_{n=0}^{\infty} \) would mean
\[ a_0, a_1, a_2, \ldots. \]
(In this case \( a_n \) would be the \( (n+1) \)-st term.)

For some sequences, it is possible to give an explicit formula for \( a_n \): this means that \( a_n \) is expressed as a function of \( n \). For instance, the sequence (1) above can be described by the explicit formula \( a_n = 2^{n-1} \).

2. Recursive definitions

An alternative way to describe a sequence is to list a few terms and to give a rule for computing the rest of the sequence. Our example (1) above can be described by the starting value \( a_1 = 1 \) and the rule \( a_{n+1} = 2a_n \) for integers \( n \geq 1 \). Starting from \( a_1 = 1 \), the rule implies that
\[ a_2 = 2a_1 = 2(1) = 2 \]
\[ a_3 = 2a_2 = 2(2) = 4 \]
\[ a_4 = 2a_3 = 2(4) = 8, \]
and so on; each term in the sequence can be computed recursively in terms of the terms previously computed. A rule such as this giving the next term in terms of earlier terms is also called a recurrence relation (or simply recurrence).

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3. Linear recursive sequences

A sequence \{a_n\} is said to satisfy the linear recurrence with coefficients \(c_k, c_{k-1}, \ldots, c_0\) if

\[ c_k a_{n+k} + c_{k-1} a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0 \]

holds for all integers \(n\) for which this makes sense. (If the sequence starts with \(a_1\), then this means for \(n \geq 1\).) The integer \(k\) is called the order of the linear recurrence.

A linear recursive sequence is a sequence of numbers \(a_1, a_2, a_3, \ldots\) satisfying some linear recurrence as above with \(c_k \neq 0\) and \(c_0 \neq 0\). For example, the sequence (1) satisfies

\[ a_{n+1} - 2a_n = 0 \]

for all integers \(n \geq 1\), so it is a linear recursive sequence satisfying a recurrence of order 1, with \(c_1 = 1\) and \(c_0 = -2\).

Requiring \(c_k \neq 0\) guarantees that the linear recurrence can be used to express \(a_{n+k}\) as a linear combination of earlier terms:

\[ a_{n+k} = -\frac{c_k}{c_0} a_{n+k-1} - \frac{c_{k-1}}{c_0} a_{n+k-2} - \cdots - \frac{c_1}{c_0} a_{n+1} - \frac{c_0}{c_0} a_n. \]

The requirement \(c_0 \neq 0\) lets one express \(a_n\) as a linear combination of later terms:

\[ a_n = -\frac{c_k}{c_0} a_{n+k} - \frac{c_{k-1}}{c_0} a_{n+k-1} - \cdots - \frac{c_1}{c_0} a_{n+1} - \frac{c_0}{c_0} a_n. \]

This lets one define \(a_0, a_{-1}\), and so on, to obtain a doubly infinite sequence

\(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\)

that now satisfies the same linear recurrence for all integers \(n\), positive or negative.

4. Characteristic polynomials

The characteristic polynomial of a linear recurrence

\[ c_k a_{n+k} + c_{k-1} a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0 \]

is defined to be the polynomial

\[ c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0. \]

For example, the characteristic polynomial of the recurrence \(a_{n+1} - 2a_n = 0\) satisfied by the sequence (1) is \(x - 2\).

Here is another example: the famous Fibonacci sequence

\[ \{F_n\}_{n=0}^\infty = 0, 1, 1, 2, 3, 5, 8, 13, \ldots \]

which can be described by the starting values \(F_0 = 0, F_1 = 1\) and the recurrence relation

\[ F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 2. \]

To find the characteristic polynomial, we first need to rewrite the recurrence relation in the form (2). The relation (3) is equivalent to

\[ F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0. \]

Rewriting it as

\[ F_{n+2} - F_{n+1} - F_n = 0 \]

shows that \(\{F_n\}\) is a linear recursive sequence satisfying a recurrence of order 2, with \(c_2 = 1\), \(c_1 = -1\), and \(c_0 = -1\). The characteristic polynomial is \(x^2 - x - 1\).
5. Ideals and Minimal Characteristic Polynomials

The same sequence can satisfy many different linear recurrences. For example, doubling (5) shows the Fibonacci sequence also satisfies

$$2F_{n+2} - 2F_{n+1} - 2F_n = 0,$$

which is a linear recurrence with characteristic polynomial $2x^2 - 2x - 2$. It also satisfies

$$F_{n+3} - F_{n+2} - F_{n+1} = 0,$$

and adding these two relations, we find that \{F_n\} also satisfies

$$F_{n+3} + F_{n+2} - 3F_{n+1} - 2F_n = 0,$$

which has characteristic polynomial $x^3 + x^2 - 3x - 2 = (x + 2)(x^2 - x - 1)$.

Now consider an arbitrary sequence \{a_n\}. Let I be the set of characteristic polynomials of all linear recurrences satisfied by \{a_n\}. Then

(a) If $f(x) \in I$ and $g(x) \in I$ then $f(x) + g(x) \in I$.
(b) If $f(x) \in I$ and $h(x)$ is any polynomial, then $h(x)f(x) \in I$.

In general, a nonempty set I of polynomials satisfying (a) and (b) is called an **ideal**.

**Fact from algebra:** Let I be an ideal of polynomials. Then either $I = \{0\}$ or else there is a unique monic polynomial $f(x) \in I$ such that

$$I = \text{the set of polynomial multiples of } f(x) = \{ h(x)f(x) \mid h(x) \text{ is a polynomial} \}.$$

(A polynomial is monic if the coefficient of the highest power of $x$ is 1.)

This fact, applied to the ideal of characteristic polynomials of a linear recursive sequence \{a_n\} shows that there is always a minimal characteristic polynomial $f(x)$, which is the monic polynomial of lowest degree in I. It is the characteristic polynomial of the lowest order non-trivial linear recurrence satisfied by \{a_n\}. The characteristic polynomial of any other linear recurrence satisfied by \{a_n\} is a polynomial multiple of $f(x)$.

The order of a linear recursive sequence \{a_n\} is defined to be the lowest order among all (nontrivial) linear recurrences satisfied by \{a_n\}. The order also equals the degree of the minimal characteristic polynomial. For example, as we showed above, \{F_n\} satisfies

$$F_{n+3} + F_{n+2} - 3F_{n+1} - 2F_n = 0,$$

but we also know that

$$F_{n+2} - F_{n+1} - F_n = 0,$$

and it is easy to show that \{F_n\} cannot satisfy a linear recurrence of order less than 2, so \{F_n\} is a linear recursive sequence of order 2, with minimal characteristic polynomial $x^2 - x - 1$.

6. The Main Theorem

**Theorem 1.** Let $f(x) = c_k x^k + \cdots + c_0$ be a polynomial with $c_k \neq 0$ and $c_0 \neq 0$. Factor $f(x)$ over the complex numbers as

$$f(x) = c_k (x - r_1)^{m_1} (x - r_2)^{m_2} \cdots (x - r_\ell)^{m_\ell},$$

where \( r_1, r_2, \ldots, r_\ell \) are distinct nonzero complex numbers, and \( m_1, m_2, \ldots, m_\ell \) are positive integers. Then a sequence \( \{a_n\} \) satisfies the linear recurrence with characteristic polynomial \( f(x) \) if and only if there exist polynomials \( g_1(n), g_2(n), \ldots, g_\ell(n) \) with \( \deg g_i \leq m_i - 1 \) for \( i = 1, 2, \ldots, \ell \) such that

\[
a_n = g_1(n)r_1^n + \cdots + g_\ell(n)r_\ell^n \quad \text{for all } n.
\]

Here is an important special case.

**Corollary 2.** Suppose in addition that \( f(x) \) has no repeated factors; in other words suppose that \( m_1 = m_2 = \cdots = m_\ell = 1 \). Then \( f(x) = c_k(x-r_1)(x-r_2)\cdots(x-r_\ell) \) where \( r_1, r_2, \ldots, r_\ell \) are distinct nonzero complex numbers (the roots of \( f \)). Then \( \{a_n\} \) satisfies the linear recurrence with characteristic polynomial \( f(x) \) if and only if there exist constants \( B_1, B_2, \ldots, B_\ell \) (not depending on \( n \)) such that

\[
a_n = B_1r_1^n + \cdots + B_\ell r_\ell^n \quad \text{for all } n.
\]

7. Example: solving a linear recurrence

Suppose we want to find an explicit formula for the sequence \( \{a_n\} \) satisfying \( a_0 = 1 \), \( a_1 = 4 \), and

\[
a_{n+2} = \frac{a_{n+1} + a_n}{2} \quad \text{for } n \geq 0.
\]

Since \( \{a_n\} \) satisfies a linear recurrence with characteristic polynomial \( x^2 - \frac{1}{2}x - \frac{1}{2} = (x-1)(x+\frac{1}{2}) \), we know that there exist constants \( A \) and \( B \) such that

\[
a_n = A(1)^n + B\left(-\frac{1}{2}\right)^n
\]

for all \( n \). The formula (7) is called the general solution to the linear recurrence (6). To find the particular solution with the correct values of \( A \) and \( B \), we use the known values of \( a_0 \) and \( a_1 \):

\[
1 = a_0 = A(1)^0 + B\left(-\frac{1}{2}\right)^0 = A + B
\]

\[
4 = a_1 = A(1)^1 + B\left(-\frac{1}{2}\right)^1 = A - B/2.
\]

Solving this system of equations yields \( A = 3 \) and \( B = -2 \). Thus the particular solution is

\[
a_n = 3 - 2\left(-\frac{1}{2}\right)^n.
\]

(As a check, one can try plugging in \( n = 0 \) or \( n = 1 \).)

8. Example: the formula for the Fibonacci sequence

As we worked out earlier, \( \{F_n\} \) satisfies a linear recurrence with characteristic polynomial \( x^2 - x - 1 \). By the quadratic formula, this factors as \( (x - \alpha)(x - \beta) \) where \( \alpha = (1 + \sqrt{5})/2 \) is the golden ratio, and \( \beta = (1 - \sqrt{5})/2 \). The main theorem implies that there are constants \( A \) and \( B \) such that

\[
F_n = A\alpha^n + B\beta^n
\]
for all \( n \). Using \( F_0 = 0 \) and \( F_1 = 1 \) we obtain
\[
0 = A + B, \quad 1 = A\alpha + B\beta.
\]
Solving for \( A \) and \( B \) yields
\[
A = \frac{1}{\alpha - \beta}, \quad B = \frac{-1}{\alpha - \beta},
\]
so
\[
F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]
\]
for all \( n \).

9. **Example: Finding a Linear Recurrence from an Explicit Formula**

Let \( a_n = (n + 2^n)F_n \), where \( \{F_n\} \) is the Fibonacci sequence. Then by the explicit formula for \( F_n \),
\[
a_n = (n + 2^n) \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right)
= \left[ \left( \frac{1}{\alpha - \beta} \right) n \right] \alpha^n + \left[ \left( \frac{-1}{\alpha - \beta} \right) n \right] \beta^n + \left( \frac{1}{\alpha - \beta} \right) (2\alpha)^n + \left( \frac{-1}{\alpha - \beta} \right) (2\beta)^n.
\]
By Theorem 1, \( \{a_n\} \) satisfies a linear recurrence with characteristic polynomial
\[
(x - \alpha)^2(x - \beta)^2(x - 2\alpha)(x - 2\beta) = (x^2 - x - 1)^2 \left[ x^2 - 2(\alpha + \beta) + 4\alpha\beta \right]
= (x^2 - x - 1)^2(x^2 - 2x - 4)
= x^6 - 4x^5 - x^4 + 12x^3 + x^2 - 10x + 4,
\]
where we have used the identity \( x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - x - 1 \) to compute \( \alpha + \beta \) and \( \alpha\beta \).
In other words,
\[
a_{n+6} - 4a_{n+5} - a_{n+4} + 12a_{n+3} + a_{n+2} - 10a_{n+1} + 4a_n = 0
\]
for all \( n \). In fact, we have found the minimal characteristic polynomial, since if the actual minimal characteristic polynomial were a proper divisor of \( (x^2 - x - 1)^2(x^2 - 2x - 4) \), then according to Theorem 1, the explicit formula for \( a_n \) would have had a different, simpler form.

10. **Inhomogeneous Recurrence Relations**

Suppose we wanted an explicit formula for a sequence \( \{a_n\} \) satisfying \( a_0 = 0 \), and
\[
a_{n+1} - 2a_n = F_n \quad \text{for } n \geq 0,
\]
where \( \{F_n\} \) is the Fibonacci sequence as usual. This is not a linear recurrence in the sense we have been talking about (because of the \( F_n \) on the right hand side instead of 0), so our usual method does not work. A recurrence of this type, linear except for a function of \( n \) on the right hand side, is called an *inhomogeneous recurrence*.

We can solve inhomogeneous recurrences explicitly when the right hand side is itself a linear recursive sequence. In our example, \( \{a_n\} \) also satisfies
\[
a_{n+2} - 2a_{n+1} = F_{n+1}
\]
and
\[
a_{n+3} - 2a_{n+2} = F_{n+2}.
\]
Subtracting (8) and (9) from (10) yields
\[ a_{n+3} - 3a_{n+2} + a_{n+1} + 2a_n = F_{n+2} - F_{n+1} - F_n = 0. \]
Thus \( \{a_n\} \) is a linear recursive sequence after all! The characteristic polynomial of this new linear recurrence is \( x^3 - 3x^2 + x + 2 = (x - 2)(x^2 - x - 1) \), so by Theorem 1, there exist constants \( A, B, C \) such that
\[ a_n = A \cdot 2^n + B \alpha^n + C \beta^n \]
for all \( n \). Now we can use \( a_0 = 0 \), and the values \( a_1 = 0 \) and \( a_2 = 1 \) obtained from (8) to determine \( A, B, C \). After some work, one finds
\[ A = 1, \quad B = -\alpha^2/(\alpha - \beta), \quad C = \beta^2/(\alpha - \beta), \]
so
\[ a_n = 2^n - F_{n+2}. \]

If \( \{x_n\} \) is any other sequence satisfying
\[ x_{n+1} - 2x_n = F_n \]
but not necessarily \( x_0 = 0 \), then subtracting (8) from (11) shows that the sequence \( \{y_n\} \) defined by \( y_n = x_n - a_n \) satisfies \( y_{n+1} - 2y_n = 0 \) for all \( n \), so \( y_n = D \cdot 2^n \) for some number \( D \). Hence the general solution of (11) has the form
\[ x_n = 2^n - F_{n+2} + D \cdot 2^n, \]
or more simply,
\[ x_n = E \cdot 2^n - F_{n+2}, \]
where \( E \) is some constant.

In general, this sort of argument proves the following.

**Theorem 3.** Let \( \{b_n\} \) be a linear recursive sequence satisfying a recurrence with characteristic polynomial \( f(x) \). Let \( g(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0 \) be a polynomial. Then every solution \( \{x_n\} \) to the inhomogeneous recurrence
\[ c_k x_{n+k} + c_{k-1} x_{n+k-1} + \cdots + c_1 x_{n+1} + c_0 x_n = b_n \]
also satisfies a linear recurrence with characteristic polynomial \( f(x) g(x) \).

Moreover, if \( \{x_n\} = \{a_n\} \) is one particular solution to (12), then all solutions have the form \( x_n = a_n + y_n \), where \( \{y_n\} \) ranges over the solutions of the linear recurrence
\[ c_k y_{n+k} + c_{k-1} y_{n+k-1} + \cdots + c_1 y_{n+1} + c_0 y_n = 0. \]

**11. The Mahler-Lech theorem**

Here is a deep theorem about linear recursive sequences:

**Theorem 4 (Mahler-Lech theorem).** Let \( \{a_n\} \) be a linear recursive sequence of complex numbers, and let \( c \) be a complex number. Then there exists a finite (possibly empty) list of arithmetic progressions \( T_1, T_2, \ldots, T_m \) and a finite (possibly empty) set \( S \) of integers such that
\[ \{ n \mid a_n = c \} = S \cup T_1 \cup T_2 \cup \cdots \cup T_m. \]

Warning: don’t try to prove this at home! This is *very* hard to prove. The proof uses “\( p \)-adic numbers.”
12. Problems

There are a lot of problems here. Just do the ones that interest you.

1. If the Fibonacci sequence is extended to a doubly infinite sequence satisfying the same linear recurrence, then what will $F_{-4}$ be? (Is it easier to do this using the recurrence, or using the explicit formula?)

2. Find the smallest degree polynomial that could be the minimal characteristic polynomial of a sequence that begins

$$2, 5, 18, 67, 250, 933, \ldots$$

Assuming that the sequence is a linear recursive sequence with this characteristic polynomial, find an explicit formula for the $n$-th term.

3. Suppose that $a_n = n^2 + 3n + 7$ for $n \geq 1$. Prove that $\{a_n\}$ is a linear recursive sequence, and find its minimal characteristic polynomial.

4. Suppose $a_1 = a_2 = a_3 = 1$, $a_4 = 3$, and $a_{n+4} = 3a_{n+2} - 2a_n$ for $n \geq 1$. Prove that $a_n = 1$ if and only if $n$ is odd or $n = 2$. (This is an instance of the Mahler-Lech theorem: for this sequence, one would take $S = \{2\}$ and $T = \{1, 3, 5, 7, \ldots \}$.)

5. Suppose $a_0 = 2$, $a_1 = 5$, and $a_{n+2} = (a_{n+1})^2(a_n)^3$ for $n \geq 0$. (This is a recurrence relation, but not a linear recurrence relation.) Find an explicit formula for $a_n$.

6. Suppose $\{a_n\}$ is a sequence such that $a_{n+2} = a_{n+1} - a_n$ for all $n \geq 1$. Given that $a_{38} = 7$ and $a_{55} = 3$, find $a_1$. (Hint: it is possible to solve this problem with very little calculation.)

7. Let $\theta$ be a fixed real number, and let $a_n = \cos(n\theta)$ for integers $n \geq 1$. Prove that $\{a_n\}$ is a linear recursive sequence, and find the minimal characteristic polynomial. (Hint: if you know the definition of $\cos x$ in terms of complex exponentials, use that. Otherwise, use the sum-to-product rule for the sum of cosines $\cos(n\theta) + \cos((n + 2)\theta)$. For most but not all $\theta$, the degree of the minimal characteristic polynomial will be 2.)

8. Give an example of a sequence that is not a linear recursive sequence, and prove that it is not one.

9. Given a finite set $S$ of positive integers, show that there exists a linear recursive sequence

$$a_1, a_2, a_3, \ldots$$

such that $\{ n \mid a_n = 0 \} = S$.

10. A student tosses a fair coin and scores one point for each head that turns up, and two points for each tail. Prove that the probability of the student scoring $n$ points at some time in a sequence of $n$ tosses is $\frac{1}{3} \left( 2 + \left( -\frac{1}{2} \right)^n \right)$.

11. Let $F_n$ denote the $n$-th Fibonacci number. Let $a_n = (F_n)^2$. Prove that $a_1, a_2, a_3, \ldots$ is a linear recursive sequence, and find its minimal characteristic polynomial.

12. Prove the “fact from algebra” mentioned above in Section 5. (Hint: if $I \neq \{0\}$, pick a nonzero polynomial in $I$ of smallest degree, and multiply it by a constant to get a monic polynomial $f(x)$. Use long division of polynomials to show that anything else in $I$ is a polynomial multiple of $f(x)$.)

13. Suppose that $a_1, a_2, \ldots$ is a linear recursive sequence. For $n \geq 1$, let $s_n = a_1 + a_2 + \cdots + a_n$. Prove that $\{s_n\}$ is a linear recursive sequence.

14. Suppose $\{a_n\}$ and $\{b_n\}$ are linear recursive sequences. Let $c_n = a_n + b_n$ and $d_n = a_nb_n$ for $n \geq 1$.

(a) Prove that $\{c_n\}$ and $\{d_n\}$ also are linear recursive sequences.
(b) Suppose that the minimal characteristic polynomials for \( \{a_n\} \) and \( \{b_n\} \) are \( x^2 - x - 2 \) and \( x^2 - 5x + 6 \), respectively. What are the possibilities for the minimal characteristic polynomials of \( \{c_n\} \) and \( \{d_n\} \)?

15. Suppose that \( \{a_n\} \) and \( \{b_n\} \) are linear recursive sequences. Prove that

\[
a_1, b_1, a_2, b_2, a_3, b_3, \ldots
\]

also is a linear recursive sequence.

16. Use the Mahler-Lech theorem to prove the following generalization.

Let \( \{a_n\} \) be a linear recursive sequence of complex numbers, and let \( p(x) \) be a polynomial. Then there exists a finite (possibly empty) list of arithmetic progressions \( T_1, T_2, \ldots, T_m \) and a finite (possibly empty) set \( S \) of integers such that

\[
\{ n \mid a_n = p(n) \} = S \cup T_1 \cup T_2 \cup \cdots \cup T_m.
\]

(Hint: let \( b_n = a_n - p(n) \).)

17. (1973 USAMO, no. 2) Let \( \{X_n\} \) and \( \{Y_n\} \) denote two sequences of integers defined as follows:

\[
X_0 = 1, X_1 = 1, X_{n+1} = X_n + 2X_{n-1} \quad (n = 1, 2, 3, \ldots),
\]

\[
Y_0 = 1, Y_1 = 7, Y_{n+1} = 2Y_n + 3Y_{n-1} \quad (n = 1, 2, 3, \ldots).
\]

Thus, the first few terms of the sequences are:

\[
X : 1, 1, 3, 5, 11, 21, \ldots,
\]

\[
Y : 1, 7, 17, 55, 161, 487, \ldots.
\]

Prove that, except for the “1,” there is no term which occurs in both sequences.

18. (1963 IMO, no. 4) Find all solutions \( x_1, x_2, x_3, x_4, x_5 \) to the system

\[
x_5 + x_2 = yx_1
\]

\[
x_1 + x_3 = yx_2
\]

\[
x_2 + x_4 = yx_3
\]

\[
x_3 + x_5 = yx_4
\]

\[
x_4 + x_1 = yx_5,
\]

where \( y \) is a parameter. (Hint: define \( x_6 = x_1, x_7 = x_2, \) etc., and find two different linear recurrences satisfied by \( \{x_n\} \).

19. (1967 IMO, no. 6) In a sports contest, there were \( m \) medals awarded on \( n \) successive days (\( n > 1 \)). On the first day, one medal and \( 1/7 \) of the remaining \( m - 1 \) medals were awarded. On the second day, two medals and \( 1/7 \) of the now remaining medals were awarded; and so on. On the \( n \)-th and last day, the remaining \( n \) medals were awarded. How many days did the contest last, and how many medals were awarded altogether?

item (1974 IMO, no. 3) Prove that the number \( \sum_{k=0}^{n} \binom{2n+1}{k+1} 2^{3k} \) is not divisible by 5 for any integer \( n \geq 0 \).

20. (1980 USAMO, no. 3) Let \( F_r = x^r \sin(rA) + y^r \sin(rB) + z^r \sin(rC) \), where \( x, y, z, A, B, C \) are real and \( A + B + C \) is an integral multiple of \( \pi \). Prove that if \( F_1 = F_2 = 0 \), then \( F_r = 0 \) for all positive integral \( r \).

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