LINEAR RECURSIVE SEQUENCES

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1. Sequences

A *sequence* is an infinite list of numbers, like

$$1, 2, 4, 8, 16, 32, \dots. \tag{1}$$

The numbers in the sequence are called its *terms*. The general form of a sequence is

$$a_1, a_2, a_3, \dots$$

where a_n is the n-th term of the sequence. In the example (1) above, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and so on.

The notations $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ are abbreviations for

$$a_1, a_2, a_3, \dots.$$

Occasionally the indexing of the terms will start with something other than 1. For example, $\{a_n\}_{n=0}^{\infty}$ would mean

$$a_0, a_1, a_2, \dots.$$

(In this case a_n would be the $(n+1)$-st term.)

For some sequences, it is possible to give an *explicit formula* for a_n: this means that a_n is expressed as a function of n. For instance, the sequence (1) above can be described by the explicit formula $a_n = 2^{n-1}$.

2. Recursive definitions

An alternative way to describe a sequence is to list a few terms and to give a rule for computing the rest of the sequence. Our example (1) above can be described by the starting value $a_1 = 1$ and the rule $a_{n+1} = 2a_n$ for integers $n \geq 1$. Starting from $a_1 = 1$, the rule implies that

$$\begin{aligned} a_2 &= 2a_1 = 2(1) = 2 \\ a_3 &= 2a_2 = 2(2) = 4 \\ a_4 &= 2a_3 = 2(4) = 8, \end{aligned}$$

and so on; each term in the sequence can be computed recursively in terms of the terms previously computed. A rule such as this giving the next term in terms of earlier terms is also called a *recurrence relation* (or simply *recurrence*).

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3. Linear recursive sequences

A sequence $\{a_n\}$ is said to satisfy the *linear recurrence* with coefficients c_k, c_{k-1}, ... , c_0 if

$$c_k a_{n+k} + c_{k-1} a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0 \tag{2}$$

holds for all integers n for which this makes sense. (If the sequence starts with a_1, then this means for $n \geq 1$.) The integer k is called the *order* of the linear recurrence.

A *linear recursive sequence* is a sequence of numbers $a_1, a_2, a_3, \ldots$ satisfying some linear recurrence as above with $c_k \neq 0$ and $c_0 \neq 0$. For example, the sequence (1) satisfies

$$a_{n+1} - 2a_n = 0$$

for all integers $n \geq 1$, so it is a linear recursive sequence satisfying a recurrence of order 1, with $c_1 = 1$ and $c_0 = -2$.

Requiring $c_k \neq 0$ guarantees that the linear recurrence can be used to express a_{n+k} as a linear combination of earlier terms:

$$a_{n+k} = -\frac{c_{k-1}}{c_k} a_{n+k-1} - \cdots - \frac{c_1}{c_k} a_{n+1} - \frac{c_0}{c_k} a_n.$$

The requirement $c_0 \neq 0$ lets one express a_n as a linear combination of *later* terms:

$$a_n = -\frac{c_k}{c_0} a_{n+k} - \frac{c_{k-1}}{c_0} a_{n+k-1} - \cdots - \frac{c_1}{c_0} a_{n+1}.$$

This lets one *define* a_0, a_{-1}, and so on, to obtain a *doubly infinite sequence*

$$\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$$

that now satisfies the same linear recurrence for *all* integers n, positive or negative.

4. Characteristic polynomials

The *characteristic polynomial* of a linear recurrence

$$c_k a_{n+k} + c_{k-1} a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0$$

is defined to be the polynomial

$$c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0.$$

For example, the characteristic polynomial of the recurrence $a_{n+1} - 2a_n = 0$ satisfied by the sequence (1) is $x - 2$.

Here is another example: the famous *Fibonacci sequence*

$$\{F_n\}_{n=0}^{\infty} = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

which can be described by the starting values $F_0 = 0$, $F_1 = 1$ and the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for all } n \geq 2. \tag{3}$$

To find the characteristic polynomial, we first need to rewrite the recurrence relation in the form (2). The relation (3) is equivalent to

$$F_{n+2} = F_{n+1} + F_n \qquad \text{for all } n \geq 0. \tag{4}$$

Rewriting it as

$$F_{n+2} - F_{n+1} - F_n = 0 \tag{5}$$

shows that $\{F_n\}$ is a linear recursive sequence satisfying a recurrence of order 2, with $c_2 = 1$, $c_1 = -1$, and $c_0 = -1$. The characteristic polynomial is $x^2 - x - 1$.

5. Ideals and minimal characteristic polynomials

The same sequence can satisfy many different linear recurrences. For example, doubling (5) shows the Fibonacci sequence also satisfies

$$2F_{n+2} - 2F_{n+1} - 2F_n = 0,$$

which is a linear recurrence with characteristic polynomial $2x^2 - 2x - 2$. It also satisfies

$$F_{n+3} - F_{n+2} - F_{n+1} = 0,$$

and adding these two relations, we find that $\{F_n\}$ also satisfies

$$F_{n+3} + F_{n+2} - 3F_{n+1} - 2F_n = 0$$

which has characteristic polynomial $x^3 + x^2 - 3x - 2 = (x+2)(x^2 - x - 1)$.

Now consider an arbitrary sequence $\{a_n\}$. Let I be the set of characteristic polynomials of *all* linear recurrences satisfied by $\{a_n\}$. Then

(a) If $f(x) \in I$ and $g(x) \in I$ then $f(x) + g(x) \in I$.
(b) If $f(x) \in I$ and $h(x)$ is any polynomial, then $h(x)f(x) \in I$.

In general, a nonempty set I of polynomials satisfying (a) and (b) is called an *ideal.*

Fact from algebra: Let I be an ideal of polynomials. Then either $I = \{0\}$ or else there is a unique monic polynomial $f(x) \in I$ such that

$$I = \text{the set of polynomial multiples of } f(x) = \{\, h(x)f(x) \mid h(x) \text{ is a polynomial} \,\}.$$

(A polynomial is *monic* if the coefficient of the highest power of x is 1.)

This fact, applied to the ideal of characteristic polynomials of a linear recursive sequence $\{a_n\}$ shows that there is always a *minimal characteristic polynomial* $f(x)$, which is the monic polynomial of lowest degree in I. It is the characteristic polynomial of the lowest order nontrivial linear recurrence satisfied by $\{a_n\}$. The characteristic polynomial of any other linear recurrence satisfied by $\{a_n\}$ is a polynomial multiple of $f(x)$.

The *order* of a linear recursive sequence $\{a_n\}$ is defined to be the lowest order among all (nontrivial) linear recurrences satisfied by $\{a_n\}$. The order also equals the degree of the minimal characteristic polynomial. For example, as we showed above, $\{F_n\}$ satisfies

$$F_{n+3} + F_{n+2} - 3F_{n+1} - 2F_n = 0,$$

but we also know that

$$F_{n+2} - F_{n+1} - F_n = 0,$$

and it is easy to show that $\{F_n\}$ cannot satisfy a linear recurrence of order less than 2, so $\{F_n\}$ is a linear recursive sequence of order 2, with minimal characteristic polynomial $x^2 - x - 1$.

6. The main theorem

Theorem 1. *Let $f(x) = c_k x^k + \cdots + c_0$ be a polynomial with $c_k \neq 0$ and $c_0 \neq 0$. Factor $f(x)$ over the complex numbers as*

$$f(x) = c_k (x - r_1)^{m_1} (x - r_2)^{m_2} \cdots (x - r_\ell)^{m_\ell},$$

where $r_1, r_2, \ldots, r_\ell$ are distinct nonzero complex numbers, and $m_1, m_2, \ldots, m_\ell$ are positive integers. Then a sequence $\{a_n\}$ satisfies the linear recurrence with characteristic polynomial $f(x)$ if and only if there exist polynomials $g_1(n)$, $g_2(n)$, ..., $g_\ell(n)$ with $\deg g_i \le m_i - 1$ for $i = 1, 2, \ldots, \ell$ such that

$$a_n = g_1(n) r_1^n + \cdots + g_\ell(n) r_\ell^n \qquad \text{for all } n.$$

Here is an important special case.

Corollary 2. *Suppose in addition that $f(x)$ has no repeated factors; in other words suppose that $m_1 = m_2 = \cdots = m_\ell = 1$. Then $f(x) = c_k(x - r_1)(x - r_2) \cdots (x - r_\ell)$ where $r_1, r_2, \ldots, r_\ell$ are distinct nonzero complex numbers (the roots of f). Then $\{a_n\}$ satisfies the linear recurrence with characteristic polynomial $f(x)$ if and only if there exist constants $B_1, B_2, \ldots, B_\ell$ (not depending on n) such that*

$$a_n = B_1 r_1^n + \cdots + B_\ell r_\ell^n \qquad \text{for all } n.$$

7. Example: solving a linear recurrence

Suppose we want to find an explicit formula for the sequence $\{a_n\}$ satisfying $a_0 = 1$, $a_1 = 4$, and

$$a_{n+2} = \frac{a_{n+1} + a_n}{2} \text{ for } n \ge 0. \tag{6}$$

Since $\{a_n\}$ satisfies a linear recurrence with characteristic polynomial $x^2 - \frac{1}{2}x - \frac{1}{2} = (x-1)(x+\frac{1}{2})$, we know that there exist constants A and B such that

$$a_n = A(1)^n + B\left(-\frac{1}{2}\right)^n \tag{7}$$

for all n. The formula (7) is called the *general solution* to the linear recurrence (6). To find the *particular solution* with the correct values of A and B, we use the known values of a_0 and a_1:

$$1 = a_0 = A(1)^0 + B\left(-\frac{1}{2}\right)^0 = A + B$$

$$4 = a_1 = A(1)^1 + B\left(-\frac{1}{2}\right)^1 = A - B/2.$$

Solving this system of equations yields $A = 3$ and $B = -2$. Thus the particular solution is

$$a_n = 3 - 2\left(-\frac{1}{2}\right)^n.$$

(As a check, one can try plugging in $n = 0$ or $n = 1$.)

8. Example: the formula for the Fibonacci sequence

As we worked out earlier, $\{F_n\}$ satisfies a linear recurrence with characteristic polynomial $x^2 - x - 1$. By the quadratic formula, this factors as $(x - \alpha)(x - \beta)$ where $\alpha = (1 + \sqrt{5})/2$ is the golden ratio, and $\beta = (1 - \sqrt{5})/2$. The main theorem implies that there are constants A and B such that

$$F_n = A\alpha^n + B\beta^n$$

for all n. Using $F_0 = 0$ and $F_1 = 1$ we obtain

$$0 = A + B, \qquad 1 = A\alpha + B\beta.$$

Solving for A and B yields $A = 1/(\alpha - \beta)$ and $B = -1/(\alpha - \beta)$, so

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for all n.

9. Example: finding a linear recurrence from an explicit formula

Let $a_n = (n + 2^n)F_n$, where $\{F_n\}$ is the Fibonacci sequence. Then by the explicit formula for F_n,

$$\begin{aligned} a_n &= (n + 2^n)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) \\ &= \left[\left(\frac{1}{\alpha - \beta}\right) n\right]\alpha^n + \left[\left(\frac{-1}{\alpha - \beta}\right) n\right]\beta^n + \left(\frac{1}{\alpha - \beta}\right)(2\alpha)^n + \left(\frac{-1}{\alpha - \beta}\right)(2\beta)^n. \end{aligned}$$

By Theorem 1, $\{a_n\}$ satisfies a linear recurrence with characteristic polynomial

$$\begin{aligned} (x-\alpha)^2(x-\beta)^2(x-2\alpha)(x-2\beta) &= (x^2 - x - 1)^2\left[x^2 - 2(\alpha+\beta) + 4\alpha\beta\right] \\ &= (x^2 - x - 1)^2(x^2 - 2x - 4) \\ &= x^6 - 4x^5 - x^4 + 12x^3 + x^2 - 10x + 4, \end{aligned}$$

where we have used the identity $x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - x - 1$ to compute $\alpha + \beta$ and $\alpha\beta$. In other words,

$$a_{n+6} - 4a_{n+5} - a_{n+4} + 12a_{n+3} + a_{n+2} - 10a_{n+1} + 4a_n = 0$$

for all n. In fact, we have found the minimal characteristic polynomial, since if the actual minimal characteristic polynomial were a proper divisor of $(x^2 - x - 1)^2(x^2 - 2x - 4)$, then according to Theorem 1, the explicit formula for a_n would have had a different, simpler form.

10. Inhomogeneous recurrence relations

Suppose we wanted an explicit formula for a sequence $\{a_n\}$ satisfying $a_0 = 0$, and

$$a_{n+1} - 2a_n = F_n \qquad \text{for } n \geq 0, \tag{8}$$

where $\{F_n\}$ is the Fibonacci sequence as usual. This is not a linear recurrence in the sense we have been talking about (because of the F_n on the right hand side instead of 0), so our usual method does not work. A recurrence of this type, linear except for a function of n on the right hand side, is called an *inhomogeneous recurrence*.

We can solve inhomogeneous recurrences explicitly when the right hand side is itself a linear recursive sequence. In our example, $\{a_n\}$ also satisfies

$$a_{n+2} - 2a_{n+1} = F_{n+1} \tag{9}$$

and

$$a_{n+3} - 2a_{n+2} = F_{n+2}. \tag{10}$$

Subtracting (8) and (9) from (10) yields

$$a_{n+3} - 3a_{n+2} + a_{n+1} + 2a_n = F_{n+2} - F_{n+1} - F_n = 0.$$

Thus $\{a_n\}$ is a linear recursive sequence after all! The characteristic polynomial of this new linear recurrence is $x^3 - 3x^2 + x + 2 = (x-2)(x^2 - x - 1)$, so by Theorem 1, there exist constants A, B, C such that

$$a_n = A \cdot 2^n + B\alpha^n + C\beta^n$$

for all n. Now we can use $a_0 = 0$, and the values $a_1 = 0$ and $a_2 = 1$ obtained from (8) to determine A, B, C. After some work, one finds $A = 1$, $B = -\alpha^2/(\alpha-\beta)$, and $C = \beta^2/(\alpha-\beta)$, so $a_n = 2^n - F_{n+2}$.

If $\{x_n\}$ is any other sequence satisfying

$$x_{n+1} - 2x_n = F_n \tag{11}$$

but not necessarily $x_0 = 0$, then subtracting (8) from (11) shows that the sequence $\{y_n\}$ defined by $y_n = x_n - a_n$ satisfies $y_{n+1} - 2y_n = 0$ for all n, so $y_n = D \cdot 2^n$ for some number D. Hence the *general solution* of (11) has the form

$$x_n = 2^n - F_{n+2} + D \cdot 2^n,$$

or more simply,

$$x_n = E \cdot 2^n - F_{n+2},$$

where E is some constant.

In general, this sort of argument proves the following.

Theorem 3. *Let $\{b_n\}$ be a linear recursive sequence satisfying a recurrence with characteristic polynomial $f(x)$. Let $g(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$ be a polynomial. Then every solution $\{x_n\}$ to the inhomogeneous recurrence*

$$c_k x_{n+k} + c_{k-1}x_{n+k-1} + \cdots + c_1 x_{n+1} + c_0 x_n = b_n \tag{12}$$

also satisfies a linear recurrence with characteristic polynomial $f(x)g(x)$.

Moreover, if $\{x_n\} = \{a_n\}$ is one particular solution to (12), then all solutions have the form $x_n = a_n + y_n$, where $\{y_n\}$ ranges over the solutions of the linear recurrence

$$c_k y_{n+k} + c_{k-1}y_{n+k-1} + \cdots + c_1 y_{n+1} + c_0 y_n = 0.$$

11. The Mahler-Lech theorem

Here is a deep theorem about linear recursive sequences:

Theorem 4 (Mahler-Lech theorem)**.** *Let $\{a_n\}$ be a linear recursive sequence of complex numbers, and let c be a complex number. Then there exists a finite (possibly empty) list of arithmetic progressions T_1, T_2, … T_m and a finite (possibly empty) set S of integers such that*

$$\{\, n \mid a_n = c \,\} = S \cup T_1 \cup T_2 \cup \cdots \cup T_m.$$

Warning: don't try to prove this at home! This is *very* hard to prove. The proof uses "p-adic numbers."

12. Problems

There are a lot of problems here. Just do the ones that interest you.

1. If the Fibonacci sequence is extended to a doubly infinite sequence satisfying the same linear recurrence, then what will F_{-4} be? (Is it easier to do this using the recurrence, or using the explicit formula?)
2. Find the smallest degree polynomial that could be the minimal characteristic polynomial of a sequence that begins
$$2, 5, 18, 67, 250, 933, \dots.$$
Assuming that the sequence *is* a linear recursive sequence with this characteristic polynomial, find an explicit formula for the n-th term.
3. Suppose that $a_n = n^2 + 3n + 7$ for $n \geq 1$. Prove that $\{a_n\}$ is a linear recursive sequence, and find its minimal characteristic polynomial.
4. Suppose $a_1 = a_2 = a_3 = 1$, $a_4 = 3$, and $a_{n+4} = 3a_{n+2} - 2a_n$ for $n \geq 1$. Prove that $a_n = 1$ if and only if n is odd or $n = 2$. (This is an instance of the Mahler-Lech theorem: for this sequence, one would take $S = \{2\}$ and $T_1 = \{1, 3, 5, 7, \dots\}$.)
5. Suppose $a_0 = 2$, $a_1 = 5$, and $a_{n+2} = (a_{n+1})^2 (a_n)^3$ for $n \geq 0$. (This is a recurrence relation, but not a linear recurrence relation.) Find an explicit formula for a_n.
6. Suppose $\{a_n\}$ is a sequence such that $a_{n+2} = a_{n+1} - a_n$ for all $n \geq 1$. Given that $a_{38} = 7$ and $a_{55} = 3$, find a_1. (Hint: it is possible to solve this problem with very little calculation.)
7. Let θ be a fixed real number, and let $a_n = \cos(n\theta)$ for integers $n \geq 1$. Prove that $\{a_n\}$ is a linear recursive sequence, and find the minimal characteristic polynomial. (Hint: if you know the definition of $\cos x$ in terms of complex exponentials, use that. Otherwise, use the sum-to-product rule for the sum of cosines $\cos(n\theta) + \cos((n+2)\theta)$. For most but not all θ, the degree of the minimal characteristic polynomial will be 2.)
8. Give an example of a sequence that is *not* a linear recursive sequence, and prove that it is not one.
9. Given a finite set S of positive integers, show that there exists a linear recursive sequence
$$a_1, a_2, a_3, \dots$$
such that $\{\, n \mid a_n = 0 \,\} = S$.
10. A student tosses a fair coin and scores one point for each head that turns up, and two points for each tail. Prove that the probability of the student scoring n points at some time in a sequence of n tosses is $\frac{1}{3}\left(2 + (-\frac{1}{2})^n\right)$.
11. Let F_n denote the n-th Fibonacci number. Let $a_n = (F_n)^2$. Prove that $a_1, a_2, a_3, \dots$ is a linear recursive sequence, and find its minimal characteristic polynomial.
12. Prove the "fact from algebra" mentioned above in Section 5. (Hint: if $I \neq \{0\}$, pick a nonzero polynomial in I of smallest degree, and multiply it by a constant to get a monic polynomial $f(x)$. Use long division of polynomials to show that anything else in I is a polynomial multiple of $f(x)$.)
13. Suppose that $a_1, a_2, \dots$ is a linear recursive sequence. For $n \geq 1$, let $s_n = a_1 + a_2 + \cdots + a_n$. Prove that $\{s_n\}$ is a linear recursive sequence.
14. Suppose $\{a_n\}$ and $\{b_n\}$ are linear recursive sequences. Let $c_n = a_n + b_n$ and $d_n = a_n b_n$ for $n \geq 1$.
    (a) Prove that $\{c_n\}$ and $\{d_n\}$ also are linear recursive sequences.

(b) Suppose that the minimal characteristic polynomials for $\{a_n\}$ and $\{b_n\}$ are x^2-x-2 and x^2-5x+6, respectively. What are the possibilities for the minimal characteristic polynomials of $\{c_n\}$ and $\{d_n\}$?

15. Suppose that $\{a_n\}$ and $\{b_n\}$ are linear recursive sequences. Prove that
$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots$$
also is a linear recursive sequence.
16. Use the Mahler-Lech theorem to prove the following generalization.
Let $\{a_n\}$ be a linear recursive sequence of complex numbers, and let $p(x)$ be a polynomial. Then there exists a finite (possibly empty) list of arithmetic progressions T_1, T_2, ... T_m and a finite (possibly empty) set S of integers such that
$$\{\, n \mid a_n = p(n) \,\} = S \cup T_1 \cup T_2 \cup \cdots \cup T_m.$$
(Hint: let $b_n = a_n - p(n)$.)
17. (1973 USAMO, no. 2) Let $\{X_n\}$ and $\{Y_n\}$ denote two sequences of integers defined as follows:
$$X_0 = 1, X_1 = 1, X_{n+1} = X_n + 2X_{n-1} \quad (n = 1, 2, 3, \ldots),$$
$$Y_0 = 1, Y_1 = 7, Y_{n+1} = 2Y_n + 3Y_{n-1} \quad (n = 1, 2, 3, \ldots).$$
Thus, the first few terms of the sequences are:
$$X : 1, 1, 3, 5, 11, 21, \ldots,$$
$$Y : 1, 7, 17, 55, 161, 487, \ldots.$$
Prove that, except for the "1," there is no term which occurs in both sequences.
18. (1963 IMO, no. 4) Find all solutions x_1, x_2, x_3, x_4, x_5 to the system
$$x_5 + x_2 = yx_1$$
$$x_1 + x_3 = yx_2$$
$$x_2 + x_4 = yx_3$$
$$x_3 + x_5 = yx_4$$
$$x_4 + x_1 = yx_5,$$
where y is a parameter. (Hint: define $x_6 = x_1$, $x_7 = x_2$, etc., and find two different linear recurrences satisfied by $\{x_n\}$.)
19. (1967 IMO, no. 6) In a sports contest, there were m medals awarded on n successive days ($n > 1$). On the first day, one medal and 1/7 of the remaining $m-1$ medals were awarded. On the second day, two medals and 1/7 of the now remaining medals were awarded; and so on. On the n-th and last day, the remaining n medals were awarded. How many days did the contest last, and how many medals were awarded altogether?
item (1974 IMO, no. 3) Prove that the number $\sum_{k=0}^{n} \binom{2n+1}{k+1} 2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.
20. (1980 USAMO, no. 3) Let $F_r = x^r \sin(rA) + y^r \sin(rB) + z^r \sin(rC)$, where x, y, z, A, B, C are real and $A + B + C$ is an integral multiple of π. Prove that if $F_1 = F_2 = 0$, then $F_r = 0$ for all positive integral r.

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