1st Bay Area Mathematical Olympiad

February 23, 1999

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive much credit.

Please label all pages that you submit for grading with your identification number in the upper right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The five problems below are arranged in roughly increasing order of difficulty. In particular, problems 4 and 5 are quite difficult. We don't expect many students to solve all the problems; indeed, solving just one problem completely is a fine achievement. We do hope, however, that you find the experience of thinking deeply about mathematics for 4 hours to be a fun and rewarding challenge. We hope that you find BAMO interesting, and that you continue to think about the problems after the exam is over. Good luck!

1. Prove that among any 12 consecutive positive integers there is at least one which is smaller than the sum of its proper divisors. (The proper divisors of a positive integer n are all positive integers other than 1 and n which divide n. For example, the proper divisors of 14 are 2 and 7.)

2. Let C be a circle in the xy-plane with center on the y-axis and passing through $A = (0, a)$ and $B = (0, b)$ with $0 < a < b$. Let P be any other point on the circle, let Q be the intersection of the line through P and A with the x-axis, and let $O = (0, 0)$. Prove that $\angle BQP = \angle BOP$.

3. A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too (see diagram). Show that no matter what the starting positions are, it is always possible to open this lock. (Only one key at a time can be switched.)

Please turn over for problems #4 and #5.

4. Finitely many cards are placed in two stacks, with more cards in the left stack than the right. Each card has one or more distinct names written on it, although different cards may share some names. For each name, we define a "shuffle" by moving every card that has this name written on it to the opposite stack. Prove that it is always possible to end up with more cards in the right stack by picking several distinct names, and doing in turn the shuffle corresponding to each name.

5. Let $ABCD$ be a cyclic quadrilateral (a quadrilateral which can be inscribed in a circle). Let E and F be variable points on the sides AB and CD, respectively, such that $AE/EB = CF/FD$. Let P be the point on the segment EF such that $PE/PF = AB/CD$. Prove that the ratio between the areas of triangle APD and BPC does not depend on the choice of E and F.

Please remember your ID number—our record keeping will use it rather than your name.

You are cordially invited to attend the **BAMO 1999 Awards Ceremony**, which will be held at the Faculty Club of University of California, Berkeley from 11–2 on Sunday, March 7. This event will include lunch (free of charge), a mathematical talk by Professor Alan Weinstein of UC Berkeley, and the awarding of over 60 prizes, worth approximately $5000 in total. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.