INVERSION IN THE PLANE
BERKELEY MATH CIRCLE

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Definition of Inversion in the Plane

Definition 1. Let $k(O,r)$ be a circle with center O and radius r. Consider a function on the plane, $I:\mathbb{R}^2 \to \mathbb{R}^2$, sending a point $X \not\equiv O$ to the point on the half line $OX^{\rightarrow}$, X_1, defined by

$$OX \cdot OX_1 = r^2.$$

Such a function I is called an *inversion of the plane* with center O and radius r (write $I(O,r)$.)

It is immediate that I is *not* defined at p. O. But if we compactify $\mathbb{R}^2$ to a sphere by adding one extra point O_∞, we could define $I(O) = O_\infty$ and $I(O_\infty) = O$.

An inversion of the plane can be equivalently described as follows. If $X \in k$, then $I(X) = X$. If X lies outside k, draw a tangent from X to k and let X_2 be the point of tangency. Drop a perpendicular X_2X_1 towards the segment OX with $X_1 \in OX$, and set $I(X) = X_1$. The case when X is inside k, $X \not\equiv O$, is treated in a reverse manner: erect a perpendicular XX_2 to OX, with $X_2 \in k$, draw the tangent to k at point X_2 and let X_1 be the intersection of this tangent with the line OX; we set $I(X) = X_1$.

Properties of Inversion

Some of the basic properties of a plane inversion $I(O,r)$ are summarized below:

- I^2 is the identity on the plane.
- If $A \not\equiv B$, and $I(A) = A_1, I(B) = B_1$, then $\triangle OAB \sim \triangle OB_1A_1$. Consequently,

$$A_1B_1 = \frac{AB \cdot r^2}{OA \cdot OB}.$$

- If l is a line with $O \in l$, then $I(l) = l$.
- If l is a line with $O \notin l$, then $I(l)$ is a circle k_1 with diameter OM_1, where $M_1 = I(M)$ for the orthogonal projection M of O onto l.
- If k_1 is a circle through O, then $I(k_1)$ is a line l: reverse the previous construction.
- If $k_1(O_1, r_1)$ is a circle not passing through O, then $I(k_1)$ is a circle k_2 defined as follows: let A and B be the points of intersection of the line OO_1 with k_1, and let $A_1 = I(A)$ and $B_1 = I(B)$; then k_2 is the circle with diameter A_1B_1. Note that the center O_1 of k_1 does *not* map to the center O_2 of k_2.

Note that two circles are perpendicular if their tangents at a point of intersection are perpendicular; following the same rule, a line and a circle will be perpendicular if the line passes through the center of the circle. In general, the angle between a line and a circle is the angle between the line and the tangent to the circle at a point of intersection with the line.

- Inversion preserves angles between figures: let F_1 and F_2 be two figures (lines, circles); then

$$\angle(F_1, F_2) = \angle(I(F_1), I(F_2)).$$

Problems

(1) Given a point A and two circles k_1 and k_2, construct a third circle k_3 so that k_3 passes through A and is tangent to k_1 and k_2.

(2) Given two points A and B and a circle k_1, construct another circle k_2 so that k_2 passes through A and B and is tangent to k_1.

(3) Given circles k_1, k_2 and k_3, construct another circle k which tangent to all three of them.

(4) Let k be a circle, and let A and B be points on k. Let s and q be any two circles tangent to k at A and B, respectively, and tangent to each other at M. Find the set traversed by the point M as s and q move in the plane and still satisfy the above conditions.

(5) Circles k_1, k_2, k_3 and k_4 are positioned in such a way that k_1 is tangent to k_2 at point A, k_2 is tangent to k_3 at point B, k_3 is tangent to k_4 at point C, and k_4 is tangent to k_1 at point D. Show that A, B, C and D are either collinear or concyclic.

(6) Circles k_1, k_2, k_3 and k_4 intersect cyclicly pairwise in points $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{C_1, C_2\}$, and $\{D_1, D_2\}$. (k_1 and k_2 intersect in A_1 and A_2, k_2 and k_3 intersect in B_1 and B_2, etc.)
   (a) Prove that if A_1, B_1, C_1, D_1 are collinear (concyclic), then A_2, B_2, C_2, D_2 are also collinear (concyclic).
   (b) Prove that if A_1, A_2, C_1, C_2 are concyclic, then B_1, B_2, D_1, D_2 are also concyclic.

(7) (Ptolemy's Theorem) Let $ABCD$ be inscribed in a circle k. Prove that the sum of the products of the opposite sides equals the product of the diagonals of $ABCD$:
$$AB \cdot DC + AD \cdot BC = AC \cdot BD.$$
Further, prove that for any four points A, B, C, D: $AB \cdot DC + AD \cdot BC \geq AC \cdot BD$. When is equality achieved?

(8) Let k_1 and k_2 be two circles, and let P be a point. Construct a circle k_0 through P so that $\angle(k_1, k_0) = \alpha$ and $\angle(k_1, k_0) = \beta$ for some given angles $\alpha, \beta \in [0, \pi)$.

(9) Given three angles $\alpha_1, \alpha_2, \alpha_3 \in [0, \pi)$ and three circles k_1, k_2, k_3, two of which do not intersect, construct a fourth circle k so that $\angle(k, k_i) = \alpha_i$ for $i = 1, 2, 3$.

(10) Construct a circle k^* so that it goes through a given point P, touches a given line l, and intersects a given circle k at a right angle.

(11) Construct a circle k which goes through a point P, and intersects given circles k_1 and k_2 at angles $45°$ and $60°$, respectively.

(12) Let $ABCD$ and $A_1B_1C_1D_1$ be two squares oriented in the same direction. Prove that AA_1, BB_1 and CC_1 are concurrent if $D \equiv D_1$.

(13) Let $ABCD$ be a quadrilateral, and let k_1, k_2, and k_3 be the circles circumscribed around $\triangle DAC$, $\triangle DCB$, and $\triangle DBA$, respectively. Prove that if $AB \cdot CD = AD \cdot BC$, then k_2 and k_3 intersect k_1 at the same angle.

(14) In the quadrilateral $ABCD$, set $\angle A + \angle C = \beta$.
   (a) If $\beta = 90°$, prove that that $(AB \cdot CD)^2 + (BC \cdot AD)^2 = (AC \cdot BD)^2$.
   (b) If $\beta = 60°$, prove that $(AB \cdot CD)^2 + (BC \cdot AD)^2 = (AC \cdot BD)^2 + AB \cdot BC \cdot CD \cdot DA$.

(15) Let k_1 and k_2 be two circles intersecting at A and B. Let t_1 and t_2 be the tangents to k_1 and k_2 at point A, and let $t_1 \cap k_2 = \{A, C\}$, $t_2 \cap k_1 = \{A, D\}$. If $E \in AB^{\rightarrow}$ such that $AE = 2AB$, prove that $ACED$ is concyclic.

(16) Let OL be the inner bisector of $\angle POQ$. A circle k passes through O and $k \cap OP^{\rightarrow} = \{A\}$, $k \cap OQ^{\rightarrow} = \{B\}$, $k \cap OL^{\rightarrow} = \{C\}$. Prove that, as k changes, the following ratio remains constant:
$$\frac{OA + OB}{OC}.$$

(17) Let a circle k^* be inside a circle k, $k^* \cap k = \emptyset$. We know that there exists a sequence of circles $k_0, k_1, ..., k_n$ such that k_i touches k, k^* and k_{i-1} for $i = 1, 2, ..., n+1$ (here $k_{n+1} = k_0$.) Show that, instead of k_1, one can start with *any* circle k'_1 tangent to both k and k^*, and still be able to fit a "ring" of n circles as above. What is n is terms of the radii of and the distance between the centers of k and k^*?

(18) Circles k_1, k_2, k_3 touch pairwise, and all touch a line l. A fourth circle k touches k_1, k_2, k_3, so that $k \cap l = \emptyset$. Find the distance from the center of k to l given that radius of k is 1.