INEQUALITIES

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1. The AM-GM inequality

The most basic arithmetic mean-geometric mean (AM-GM) inequality states simply that if x and y are nonnegative real numbers, then $(x+y)/2 \geq \sqrt{xy}$, with equality if and only if $x = y$. The last phrase "with equality..." means two things: first, if $x = y \geq 0$, then $(x+y)/2 = \sqrt{xy}$ (obvious); and second, if $(x+y)/2 = \sqrt{xy}$ for some $x, y \geq 0$, then $x = y$. It follows that if $x, y \geq 0$ and $x \neq y$, then inequality is strict: $(x+y)/2 > \sqrt{xy}$.

Here's a one-line proof of the AM-GM inequality for two variables:

$$\frac{x+y}{2} - \sqrt{xy} = \frac{1}{2}\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0.$$

The AM-GM inequality generalizes to n nonnegative numbers.

AM-GM inequality:
If $x_1, \ldots, x_n \geq 0$, then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

2. The power mean inequality

Fix $x_1, \ldots, x_n \geq 0$. For $r \neq 0$ (assume $r > 0$ if some x_i are zero), the *r-th power mean* P_r of $x_1, \ldots, x_n$ is defined to be the r-th root of the average of the r-th powers of $x_1, \ldots, x_n$:

$$P_r := \left(\frac{x_1^r + \cdots + x_n^r}{n}\right)^{1/r}.$$

This formula yields nonsense if $r = 0$, but there is a natural way to define P_0 too: it is simply defined to be the geometric mean[1]:

$$P_0 := \sqrt[n]{x_1 x_2 \ldots x_n}.$$

One also defines

$$P_\infty = \max\{x_1, \ldots, x_n\}$$

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[1]The reason for this convention is that when r is very small but nonzero the value of P_r is very close to the geometric mean, and can be made as close as desired by taking r sufficiently close to 0. In other words, for those of you who have studied limits before, one can prove that

$$\lim_{r \to 0} P_r = \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Another way of saying this, for those of you who know what a continuous function is, is that the only choice for P_0 that will make P_r depend continuously on r is to choose P_0 to be the geometric mean.

since when r is very large, P_r is a good approximation to the largest of $x_1, \dots, x_n$. For a similar reason one uses the notation

$$P_{-\infty} = \min\{x_1, \dots, x_n\}.$$

Here are some examples:

$$P_1 = \frac{x_1 + \cdots + x_n}{n}$$

is the arithmetic mean,

$$P_2 = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}$$

is sometimes called the *root mean square*. For $x_1, \dots, x_n > 0$,

$$P_{-1} = \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$$

is called the *harmonic mean*.

Power mean inequality:
Let $x_1, \dots, x_n \geq 0$. Suppose $r > s$ (and $s \geq 0$ if any of the x_i are zero). Then $P_r \geq P_s$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.

The power mean inequality holds even if $r = \infty$ or $s = -\infty$, provided that we use the definitions of P_∞ and $P_{-\infty}$ above, and the convention that $\infty > r > -\infty$ for all numbers r.

Here are some special cases of the power mean inequality:

- $P_1 \geq P_0$ (the AM-GM inequality).
- $P_0 \geq P_{-1}$ (the GM-HM inequality — HM is for "harmonic mean").
- $P_1 \geq P_{-1}$ (the AM-HM inequality).

3. Convex functions

A function $f(x)$ is *convex* if for any real numbers $a < b$, each point (c, d) on the line segment joining $(a, f(a))$ and $(b, f(b))$ lies above or at the point $(c, f(c))$ on the graph of f with the same x-coordinate.

Algebraically, this condition says that

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b). \tag{1}$$

whenever $a < b$ and for all $t \in [0, 1]$. (The left hand side represents the height of the graph of the function above the x-value $x = (1-t)a + tb$ which is a fraction t of the way from a to b, and the right hand side represents the height of the line segment above the same x-value.)

Those who know what a convex set in geometry is can interpret the condition as saying that the set $S = \{(x, y) : y \geq f(x)\}$ of points above the graph of f is a convex set. Loosely speaking, this will hold if the graph of f curves in the shape of a smile instead of a frown. For example, the function $f(x) = x^2$ is convex, as is $f(x) = x^n$ for any positive *even* integer.

One can also speak of a function $f(x)$ being convex *on an interval* I. This means that the condition (1) above holds at least when $a, b \in I$ (and $a < b$ and $t \in [0, 1]$). For example, one can show that $f(x) = x^3$ is convex on $[0, \infty)$, and that $f(x) = \sin x$ is convex on $(-\pi, 0)$.

Finally one says that a function $f(x)$ on an interval I is *strictly convex*, if

$$f((1-t)a + tb) < (1-t)f(a) + tf(b)$$