Geometric Combinatorics

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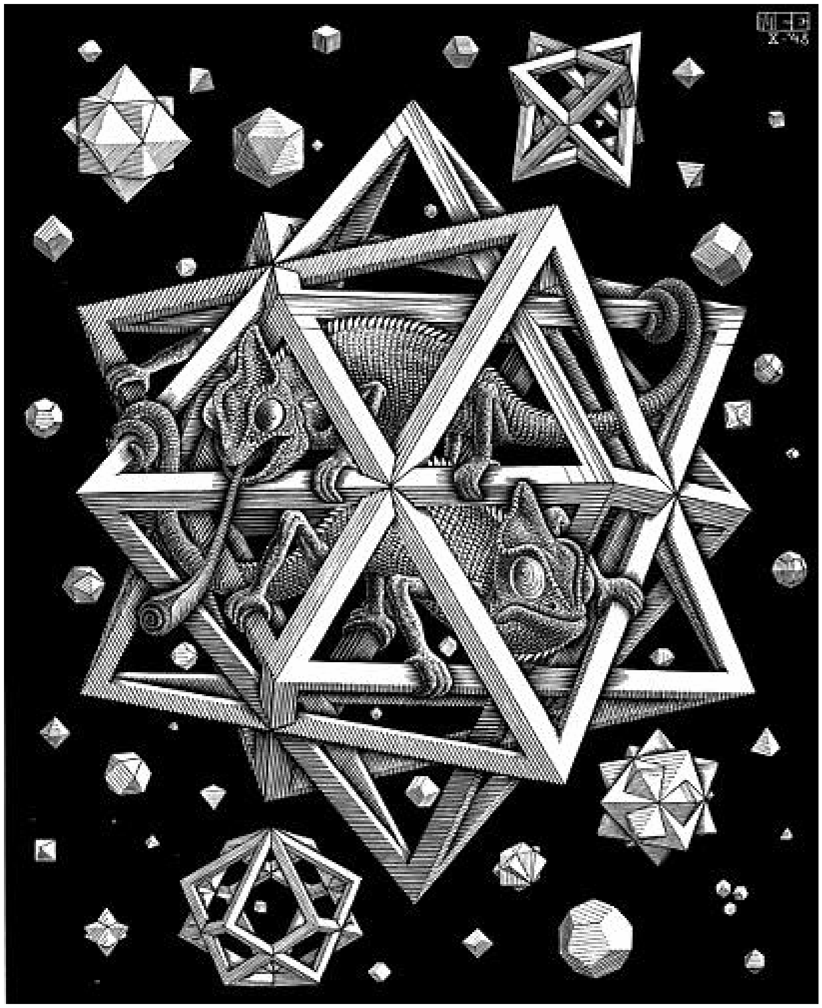

Geometric Combinatorics is a relatively new and rapidly growing branch of mathematics. It deals with geometric objects described by a finite set of building blocks, for example, bounded polyhedra and the convex hulls of finite sets of points. Other examples include arrangements and intersections of various geometric objects. Typically, problems in this area are concerned with finding bounds on a number of points or geometric figures that satisfy some conditions, or make a given configuration “optimal” in some sense.

Geometric combinatorics has many connections to linear algebra, discrete mathematics, mathematical analysis, and topology, and it has applications to economics, game theory, and biology, to name just a few.

Problems encountered within geometric combinatorics come in various forms; some are easy to state. Nevertheless, there are lots of problems that are extremely hard to solve, including a great many that remain open despite the efforts of some leading mathematicians.

The following definitions, theorems, and problems should give you some flavor of this branch of mathematics.

A *convex* planar figure is the intersection of a number (finite or infinite) of half-planes. The intersection of a finite number of half-planes is a *convex polygon*. Convex 3-d figures and convex polyhedra are defined similarly (replacing half-planes with half-spaces). Equivalently, a figure F is convex if for every two points A and B of F, F contains the entire line segment AB.

If S is any set of points, the *convex hull* of S (denoted by *conv*(*S*)) is the smallest convex figure containing S (clearly, conv(S) is the intersection of all convex sets containing S).

A *convex combination* of points $a_{1,}, a_2, \ldots, a_n$ is a linear combination $\sum_{i=1}^{n} \lambda_i a_i$ in which the coefficients λ_i are non-negative and sum to 1.

THEOREM (Carathéodory's Theorem). For S in $\mathbb{R}^n$, each point of conv(S) is a convex combination of at most n + 1 points of S.

THEOREM (Radon's Lemma). Let S be a set of size n + 2 or greater in $\mathbb{R}^n$. Then S can be partitioned in two sets R and B (*red* and *blue*) such that $conv(R) \cap conv(B) \neq \Lambda$.

THEOREM (Helly's Theorem). Suppose $S = \{ S_1, S_2, ..., S_m \}$ is a set of convex sets in $\mathbb{R}^n$, such that every subset of S of size n + 1 has non-empty intersection. Then $\bigcap_{i=1}^{m} S_i$ is non-empty.

PROBLEMS

1. Prove that if each three of n points in a plane can be enclosed in a circle of radius 1, then all n points can be enclosed in such a circle.

2. Given 7 lines in a plane, if no two of them are parallel to one another prove that there exists a pair of lines with the angle between them less than 26°.

3. How many acute inner angles can a convex *n*-gon have?

4. (a) Given $n = 4$ points in a plane. Prove that we can choose 3 out of these points so that they form a triangle with one angle at most $\alpha = 45°$. Also prove that there exists such a configuration of 4 points in a plane that every 3 of them form a triangle with each angle at least $45°$.

(b) Same as in part (a), except that $n = 5$ and $\alpha = 36°$.

(c) Same as in part (a), except that $n = 6$ and $\alpha = 30°$.

(d) Prove that for any n points in a plane ($n \geq 3$), it is possible to choose 3 points so that they form a triangle such that one of its angles is no larger than $\frac{180°}{n}$. Prove also that there exists such a configuration of n points that in any triangle formed by 3 of these points each angle is at least $\frac{180°}{n}$.

5. Let $M_1, M_2, \ldots, M_n$ be points. Find all possible values of n such that all these points can be positioned in such a way that none of the angles $\angle M_i M_j M_k$ is obtuse.

(a) assume that all these points lie in a plane;

(b) assume that the points can be placed anywhere in 3-d space.

6. Same as problem 5, except that now all the angles $\angle M_i M_j M_k$ must be acute.

7. Same as problem 5, except that now we require all the triangles $M_i M_j M_k$ be right triangles.

8. Same as problem 5, but all the triangles $M_i M_j M_k$ are required to be obtuse.

9. Prove that

(a) any convex polygon of area 1 can be covered by a parallelogram of area 2;

(b) a triangle of area 1 cannot be covered by a parallelogram of area less than 2.

10. Let M be a convex polygon with perimeter P and area S. Prove that there exists a circle of radius $\frac{S}{P}$ with all points in M.

11. (a) Prove that of all triangles of a given perimeter P, an equilateral triangle has the largest area.

(b) Prove that of all the convex quadrilaterals of a given perimeter P, a square has the largest area.

(c) Prove that of all the convex n-gons of a given perimeter P, a regular n-gon has the largest area.

12. Let k be a unit circle. Find the largest number n of unit circles $k_1, k_2, \dots, k_n$ such that k_i touches k for every $i = 1, 2, \dots, n$, and

a. k_i and k_j do not intersect for any $i \neq j$, $1 \le i, j \le n$;
b. the interior of k_i does not contain the center of k_j for any $i \neq j$, $1 \le i, j \le n$.

13. Let s be a square of side 1. Find the largest number n of squares $s_1, s_2, \dots, s_n$ such that s_i is a square of side 1 and it touches s for every $i = 1, 2, \dots, n$, and s_i, s_j do not intersect for any $i \neq j$, $1 \le i, j \le n$.

14. Let F be a planar figure, and let $h(F)$ denote the maximal number of copies of F obtained from F by translation and such that every copy touches F and no pair of these copies intersect each other.

(a) Find $h(T)$ where T is a triangle.
(b) Find $h(S)$ where S is a square.

15. Let k be a circle of radius $2R$. Find the least number n of circles $k_1, k_2, \dots, k_n$ of radius R each so that k can be covered by $k_1, k_2, \dots, k_n$.

16. Let k be a unit circle, and let $k_1, k_2, \dots$ be circles of radii less than 1. Let n be the minimal number such that $k_1, k_2, \dots, k_n$ cover k. Prove that

(a) $n \ge 3$;
(b) if the radii of $k_1, k_2, \dots$ are less than $\frac{\sqrt{3}}{2}$, then $n \ge 4$;
(c) if all the radii are less than $\frac{\sqrt{2}}{2}$, then $n \ge 5$.

Is it possible to improve the estimates in parts (a), (b), and (c)?

17. Let k be a unit circle, and let $k_1, k_2, \dots$ be circles of radii $r_1, r_2, \dots$. Prove that

(a) If $r_i > \frac{1}{2}$ $(i = 1,2,\dots)$, then it is not possible to place 2 of these circles k_i, k_j inside k in such a way that k_i and k_j do not intersect.
(b) If $r_i > 2\sqrt{3} - 3$, then it is not possible to place 3 circles k_i, k_j, k_m inside k in such a way that no two of k_i, k_j, k_m intersect each other.

18. Let M be a convex polygon and suppose that M is not a parallelogram. Prove that M can be placed inside a triangle obtained by producing 3 of its sides.

19. Given n parallel line segments in the same plane, prove that if there exists a line that intersects each three of them, then there is a line intersecting all these line segments.

20. Let A, B be points in an annulus bounded by concentric circles with radii $r \leq R$, and suppose that $AB = 1$. Find the smallest possible value of $(\angle AOB)^\circ$.

21. (a) Find R_n, the radius of the smallest circle k such that it is possible to place n points inside k (or on k) in such a way that one of the points is in the center of k, and the distance between any two of these points is at least 1.

(b) How many points can be placed inside (or on) a circle of radius 2 in such a way that one of these points is in the center of the circle, and the distance between any two of the points is at least 1?

22. Suppose there is a non-transparent spherical planet P whose diameter is d. Is it possible to place 8 observation stations on the surface of the planet in such a way that any object approaching the planet would be seen by at least 2 of these stations when the object is a distance d above the planet's surface?

23. An art gallery is in a polygonal room (with finitely many sides, and possibly non-convex) and there is one painting hung on each wall. Suppose that for any 3 paintings, there is a point in the gallery where those 3 paintings are visible. Is there always a point of the gallery from which all paintings are visible?

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SOME OPEN PROBLEMS

I. Throw k points down in the unit square and find the area of the largest convex set in the square containing none of the k points. Let $f(k)$ be the minimum (of the largest areas) over all sets of k points. Find good upper and lower bounds on $f(k)$. (For $k = 3$ it is known that $\frac{1}{3} \leq f(3) \leq \frac{\sqrt{2}}{4}$.)

II. What is the largest area that an n-gon of unit diameter can have?

III. Can every set of 8 points in the plane be partitioned to form 2 triangles and a line segment so that the segment cuts the interior of both triangles? (Notice: this is a Radon relative).