Counting Things

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Abstract

We present here various strategies for counting things. Usually, the "things" are patterns, or arrangements. For example, "How many ways can you choose 3 balls from a collection of 6 balls?"

1 General Strategies

Following this initial section on general strategies every section will begin with a short set of problems whose solution illustrates a different technique for counting things. The solutions appear in another section later in the document. The best way to read this document is to look at the problems and try to solve them yourself before going on to read the solutions. At the end (see Section 10) is a set of miscellaneous problems without solutions.

Especially at first, when you encounter a problem, you will have no idea how to proceed. Here are some general methods. Keep in mind that many problems require a combination of these methods.

- Solve a few tiny problems of the same sort and see if you can find a pattern. For example, if you don't know anything about combinations and you are asked to find the number of ways to choose six numbers from a lottery ticket with 51 numbers on it, try solving the problem for "lotteries" where you choose $1, 2,$ or 3 numbers from collections of $1, 2, 3, 4,$ or 5 possibilities. See Section 2.
- Can you break the problem into parts? In other words, if the things you are counting fall into distinct classes with no overlap, you can count the things in each class and add the results together. For example, suppose you are counting paths through a city, but you notice that every path has to go through one of two intersections. If you count the paths through each intersection somehow, you can add those numbers together to obtain the total count. See Section 3.
- If the items you are counting have independent parts, you can count the number of each kind of part and multiply the results. For example, how many license plates are there that begin with a letter of the alphabet and are followed by a 6-digit number? Well, there are 26 ways to choose the letter, and there are a million 6-digit numbers, so there are $26 \times 1000000 = 26000000$ possible license plates. See Section 3.
- There are standard formulas for counting combinations and permutations that will be discussed later in this paper. See Section 4.
- Sometimes it is easier to count the things you do *not* want. For example, suppose the problem is to count all the 5-digit numbers that contain at least one 7. This is a mess since the number can contain $1, 2, 3, 4,$ or 5 7s, but you know there are 100000 total numbers, and it is easy to count the number that have no 7s in them (there are $9^5 = 59049$ of these, so there are $100000 - 59049 = 40951$ numbers containing at least one 7. See section 5.
- Sometimes problems that seem totally different are actually equivalent. For example, problems 36 and 38 are the same, since counting the routes through the city is equivalent to dividing the movements of one block south into each of the possible north-south streets. In other words, problem 36 is a special case of problem 38 where $k = 6$ and $n = 9$.

2 Organization (Discussion: see Section 6)

1. How many ways can you choose 1 thing f