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Graph Theory Outline

1 Basic terminology: graph, psuedograph, degree, regular, k-regular, connected, connected component, planar, directed, bipartite, tree, forest, walk, trail, path, circuit, cycle, Hamilton path/cycle, Euler trail/circuit, isomorphism.

2 Fundamental and easy-to-prove theorems

(a) *Handshake Lemma*: The sum of the degrees of the vertices equals twice the number of edges; as a corollary, if v is odd, one of the vertices has even degree.

(b) A graph is bipartite if and only if it has no odd cycles.

(c) For connected graphs, $e \geq v-1$, with equality holding for trees. For a forest with k connected components, $e = v - k$.

(d) If $e \geq v$, then the graph has a cycle.

3 Fundamental and harder-to-prove theorems

(a) A connected graph has an Eulerian path if and only if it has either zero or two vertices with odd degree.

(b) A connected graph has an Eulerian circuit if and only if all vertices have even degree.

(c) If the sum of the degrees for each pair of vertices is $\geq v - 1$, the graph has a Hamilton path.

(d) A directed K_n (also known as a **tournament**) always has a (directed) Hamiltonian path.

(e) For a connected planar graph, $v - e + r = 2$, where r denotes the number of regions (including the unbounded region) that the graph divides the plane into.

Problems involving the fundamental ideas

1 Show that every graph contains two vertices of equal degree.

2 Given six people, show that either three are mutual friends, or three are complete strangers to one another. (Assume that "friendship" is mutual; i.e., if you are my friend then I must be your friend.)

3 Seventeen people are at a party. It turns out that for each pair of people present, exactly one of the following statements is always true: "They haven't met," "They are good friends," or "They hate each other." Prove that there must be a trio (3) of people, all of whom are either mutual strangers, mutual good friends, or mutual enemies.

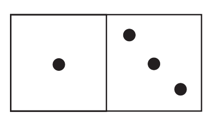
4 Complete the sentence, with proof: A connected, 2-regular graph is a ________.

5 Show that if a graph has v vertices, each of degree at least $v/2$, then this graph is connected. In fact, show that it is Hamiltonian.

6 How many edges must a graph with n vertices have in order to guarantee that it is connected?

7 A large house contains a television set in each room that has an odd number of doors. There is only one entrance to this house. Show that it is always possible to enter this house and get to a room with a television set.

8 A domino consists of two squares, each of which is marked with 0, 1, 2, 3, 4, 5, or 6 dots. Here is one example.



Verify that there are 28 different dominos. Is it possible to arrange them all in a circle so that the adjacent halves of neighboring dominos show the same number?

9 Is it possible for a knight to travel around a standard 8×8 chessboard, starting and ending at the same square, while making every single possible move that a knight can make on the chessboard, *exactly once*? We consider a move to be completed if it occurs in either direction.

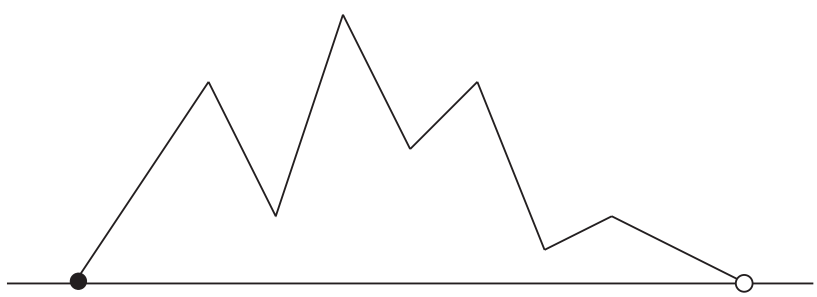
10 (USAMO 1989) The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

11 An *n-cube* is defined intuitively to be the graph you get if you try to build an n-dimensional cube out of wire. More rigorously, it is a graph with 2^n vertices labeled by the n-digit binary numbers, with two vertices joined by an edge if the binary digits differ by exactly one digit. Show that for every $n \geq 1$, the n-cube has a Hamiltonian cycle.

12 Even if a graph is not planar, it is still possible to "embed" it on a surface. This concept can be made more rigorous, but should be intuitively clear. For example, K_5 is not planar, but it can be embedded on a *torus*! Show how this can be done.

13 If you place the digits 0,1,1,0 clockwise on a circle, it is possible to read any two-digit binary number from 00 to 11 by starting at a certain digit and then reading clockwise. Is it possible to do this in general?

14 *The Two Men of Tibet*. Two men are located at opposite ends of a mountain range, at the same elevation. If the mountain range never drops below this starting elevation, is it possible for the two men to walk along the mountain range and reach each other's starting place, while always staying at the same elevation? Here is an example of a "mountain range." Without loss of generality, it is "piecewise linear," i.e., composed of straight line pieces. The starting positions of the two men is indicated by two dots.



15 A rectangle is tiled with smaller rectangles, each of which has at least one side of integral length. Prove that the tiled rectangle also must have at least one side of integral length.