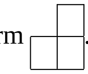
Berkeley Math Circle
BAMO 2006 Preparation – Ivan Matić

1. One cell is taken out from the talbe $2^k \times 2^k$. Prove that the rest of the board can be covered by the figures of the form .

2. Prove that the number
$$1^{2005} + 2^{2005} + 3^{2005} + \cdots + 2004^{2005}$$
is not divisible by 2006.

3. For every $n = 0, 1, 2, \ldots, 2006$ we have defined the number $A_n = 2^{3n} + 3^{6n+2} + 5^{6n+2}$. Find the greatest common divisor of the numbers $A_0, A_1, A_2, \ldots, A_2006$.

4. Find all integers x, y, z such that $x^5 + y^5 + z^5 = 2006$.

5. Find all functions f from the reals to the reals such that
$$f(f(x) + y) = 2x + f(f(y) - x)$$
for all real x, y.

6. Prove that every sequence of $mn + 1$ different natural numbers contains an increasing subsequence of $n + 1$ numbers or decreasing subsequence of $m + 1$ numbers.

7. Six arbitrary points are chosen in a rectangle 3×4. Prove that there are two points among them whose distance is at most $\sqrt{5}$.

8. There are 100 prisoners in a prison. A warden has decided to play the following game: He will put the prisoners in a row one after the other and he will put a hat ne the head of each of the prisoners. Each hat is red, green or blue. Each prisoner is allowed to say only one word- the color of his hat. If he says the correct color, he will be free; otherwise he will be killed. The first prisoner in the row can't see any of the hats; the second can see only the hat on the head of the first prisoner, ..., the last prisoner in the row can see all hats except his own.

   Before the game has started the prisoners had time to make a strategy that will safe as many of them as possible. What is the maximal number of prisoners that can be always saved?

9. Given a graph with n vertices and q edges numbered $1, \ldots, q$, show that there exists a chain of m edges, $m \geq \frac{2q}{n}$, each two consecutive edges having a common vertex, arranged monotonically with respect to the numbering.

10. More than a half of the faces of a polyhedron are colored in blue, but no two adjacent faces are blue (faces are adjacent if they share an edge). Prove that a sphere can't be inscribed in such a polyhedron.

11. Let $ABCD$ be a rectangle and E the foot of perpendicular from B to AC. If F and G are midpoints of CD and AE, respectively, prove that $\measuredangle BGF = 90^\circ$.

12. Let ABC be a triangle, and let P be a point inside it such that $\measuredangle PAC = \measuredangle PBC$. The perpendiculars from P to BC and CA meet these lines at L and M, respectively, and D is the midpoint of AB. Prove that $DL = DM$.

13. A circle with center O passes through points A and C and intersects the sides AB and BC of the triangle ABC at points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at two distinct points B and M. Prove that $\measuredangle OMB = 90^\circ$.

14. A cyclic quadrilateral $ABCD$ is given. The lines AD and BC intersect at E, with C between B and E; the diagonals AC and BD intersect at F. Let M be the midpoint of the side CD, and let $N \neq M$ be a point on the circumcircle of the triangle ABM such that $AN/BN = AM/BM$. Prove that the points E, F, and N are collinear.

15. Two players A and B play the following game on an infinite chessboard. Initially, all cells are empty. Player A starts the game and each his move consists of writing the letter X in some empty cell of the board. After his move, player B writes the letter O in some other cell, etc. The winner is the player who manages to put his sign in

    (a) all cells of some of the rectangles 1×5, 5×1 or all cells of some 2×2 square.

    (b) 11 cells that are consecutive and lie on some of the horizontal, vertical or diagonal line.

    Prove that no player has a winning strategy.