Finite Geometries

April 18, 2006

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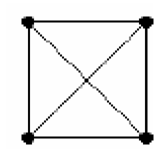
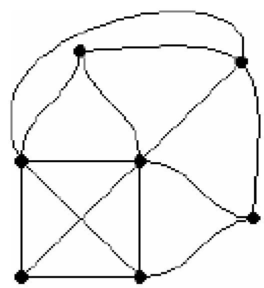
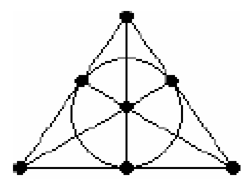

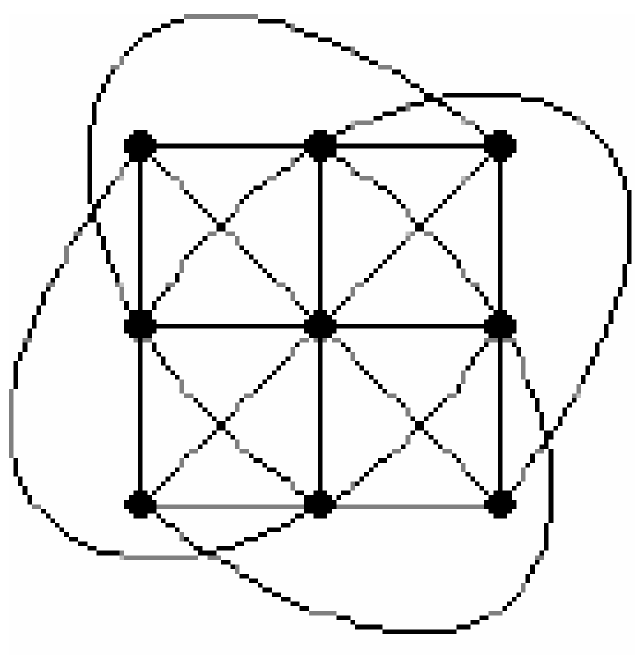

1. Introduction.

A *space* $S = (P, L)$ is a system of points P and lines L such that every line is a subset of P, and certain conditions or *axioms* are satisfied. We can use the axioms to prove some additional properties of the space; these additional properties are called *theorems*.

Any system of consistent axioms gives rise to a geometry. Some of these are more interesting than others, but they are all logically valid. The type of geometry one uses depends on the application one has in mind. Let us consider some examples.

1.1 *Three-Point Geometry*

Axioms:
- 3p–1. There exist exactly three distinct points in the geometry.
- 3p-2. Two distinct points are on exactly one line.
- 3p-3. Not all the points of the geometry are on the same line.
- 3p-4. Two distinct lines are on at least one point.

Theorems:
1. Two distinct lines are on exactly one point.
2. The three-point geometry has exactly three lines.

1.2 *Four-Line Geometry*

Axioms:
- 4l-1. There exist exactly four lines.
- 4l-2. Any two distinct lines have exactly one point on both of them.
- 4l-3. Each point is on exactly two lines.

Theorems:
1. The four-line geometry has exactly six points.
2. Each line of the four-line geometry has exactly three points on it.

1.3 *Fano's Geometry*

Axioms:
- F-1. There exists at least one line.
- F-2. Every line of the geometry has exactly three points on it.
- F-3. Not all points of the geometry are on the same line.
- F-4. For two distinct points, there exists exactly one line on both of them.
- F-5. Every two lines have at least one point on both of them.

Theorems:
1. Every two lines have exactly one point in common.
2. The geometry has exactly seven points and seven lines.
3. Each point lies on exactly three lines.
4. The lines through any one point of the geometry contain all the points of the geometry.

1.4 *Young's Geometry*

Axioms:
- Y-1. There exists at least one line.
- Y-2. Every line of the geometry has exactly three points on it.

Y-3. Not all points of the geometry are on the same line.
Y-4. For two distinct points, there exists exactly one line on both of them.
Y-5. If a point does not lie on a given line, then there exists exactly one line on that point that does not intersect the given line.

2. Near-linear and linear spaces.

Axioms: NL1 Any line has at least two points.
NL2 Two points are on at most one line.

If a space satisfies these 2 axioms, it is a near-linear space.
Examples of near-linear spaces:

1. Let P be the set of points of Euclidean 3-space, and let L be the set of all usual lines. Then (P, L) is a near-linear space.
2. Let P be as above but let L be the set of all planes in 3-space. This is not a near-linear space. (Why?)
3. Let P = {1, 2, 3, 4, 5, 6} and L = {{1, 2, 3}, {2, 4}, {3, 4, 5}, {1, 4}}. Then (P, L) is a near-linear space.
4. P = {1, 2, 3}, L = {{1}, {2, 3}}. Is (P, L) a near-linear space? Why?
5. P = {1, 2, 3}, L = {{1, 2, 3}, {1, 2}}. Is (P, L) a near-linear space?

Now, let's consider a slightly different system of axioms.

Axioms: L1 Any line has at least two points.
L2 Two points are on precisely one line.

Which of the spaces above are linear spaces?

3. Dimension of a (near-) linear space.

A subspace of a space (P, L) is a set (X, M) where X is a set of points of P such that whenever *p* and *q* are points of X that are on a line *pq*, then all points on this line are in X, and M consists of all the lines of L made of points in X. The empty set, a single point, a single line, and the entire space are always subspaces of a given space. The intersection of any number of subspaces is a subspace. (Why?)
If X is any subset of P, the *closure* of X, denoted by $\langle X \rangle$, is the smallest subspace containing X. Clearly, the closure of X is the intersection of all subspaces containing X. If $\langle X \rangle$ = R, we say that X *generates* R.

An *independent* set X is a set of points such that for each $x \in X$, $x \notin \langle X - \{x\} \rangle$.
A *basis* of a (near-) linear space S is an independent subset of the points of S which generates S.

The *dimension* of a (near-) linear space S is one less than the number of elements of its smallest basis.

What are the dimensions of all the (near-) linear spaces above?

Next two sections are concerned with particularly interesting examples of linear spaces of dimension two.

4. Finite Projective Planes.

Axioms: (F)PP-1. Every two distinct points determine a unique line.
(F)PP-2. Every two distinct lines meet at a unique point.
(F)PP-3. There exist four points, no three of which are on the same line.
FPP-4. There exist only finitely many points.

It is not hard to prove that a FPP contains $n^2 + n + 1$ points; every line has $n + 1$ points, every point is on $n + 1$ lines, and n is at least 2. (See problems 7 – 11).

This number n is called *the order* of the FPP.

5. Finite Affine Planes.

Axioms: (F)AP-1. Every two distinct points determine a unique line.
(F)AP-2. Any point not on a line l is on precisely one line missing the line l.
(F)AP-3. There exist four points no three of which are on the same line.
FAP-4. There exist only finitely many points

If ℓ_1 *and* ℓ_2 are lines of an affine plane and $\ell_1 = \ell_2$ or ℓ_1 *and* ℓ_2 do not meet, we say that they are *parallel*.

Theorems.

1. If two lines are parallel to a third one, then they are parallel to each other.
2. All lines in a FAP have the same number of points, and all points are on the same number of lines.
3. There exists an integer $k > 0$ such that:
(i) the total number of points is k^2;
(ii) each line is parallel to k lines;
(iii) the total number of lines is $k^2 + k$;
(iv) each line meets k^2 other lines;
(v) there are $k+1$ parallel classes of lines.

This number k is called *the order* of the finite affine plane.

If we have a FAP with the set of points P and the set of lines L, we can use it to construct a FPP in the following way. For every class of parallel lines in L, we'll add a new point to P, and we'll assume that this *ideal* point belongs to each of these

parallel lines. We will also add one new line consisting of all the ideal points to L.
Can you prove that the system obtained is indeed a FPP?

Problems.

1. A city bus network runs in the following fashion:
(i) one can get from any bus stop to any other without transferring;
(ii) for any pair of routes there is one and only one bus stop where one can transfer from one route to the other;
(iii) there are exactly three bus stops on each route.
How many bus routes are there in town?

2. For Young's geometry, do the following.
(a) Prove that the geometry includes at least 9 points.
(b) Find the exact number of points in the geometry.
(c) Find the exact number of lines in the geometry.
(d) Prove that two lines parallel to a third line are parallel to each other.

3. Find an example of a near-linear space with an infinite number of points and a finite number of lines.

4. Find a finite near-linear space with dimension 3.

5. Find a near-linear space of dimension 3 which contains a proper subspace of dimension 3.

6. Prove that on a projective plane there exist 4 lines, no three of which pass through the same point.

7. Let L_1 and L_2 be two distinct lines of a projective plane. Prove that there exists a one-to-one correspondence between all the points on L_1, and all the points on L_2.

8. Let P_1 and P_2 be two distinct points of a projective plane. Prove that there exists a one-to-one correspondence between all the lines through P_1 and all the lines through P_2.

9. If P is a point, and L is a line of a projective plane, prove that there exists a one-to-one correspondence between all the points on L and all the lines through P.

10. If a projective plane has a finite number of points, prove that there exists an integer $n > 0$ such that
(i) the total number of points of the plane is $n^2 + n + 1$;
(ii) the total number of lines of the plane is $n^2 + n + 1$;
(iii) every line contains $n + 1$ points;
(iv) every point is on $n + 1$ lines.

(Such a projective plane is said to be *of order n*).

11. What is the smallest number of points on a projective plane?

12. In a certain city, there are exactly 57 bus routes. It is known that
(i) one can get from any bus stop to any other without transferring;
(ii) for any pair of routes there is exactly one bus stop where one can transfer from one route to the other;
(iii) every route has at least 3 stops.
How many stops are there on every bus route?

13. A *triangle* in a finite affine plane is a set of 3 points not belonging to the same line. Prove that an affine plane with k^2 points has exactly $k^3(k-1)^2(k+1)/6$ triangles.

14. At a party, assume that no boy dances with every girl but each girl dances with at least one boy. Prove that there are two couples g_1b_1 and g_2b_2 which dance whereas b_1 does not dance with g_2 nor does g_1 dance with b_2.

15. In a round-robin tournament with n players $P_1, P_2, \ldots, P_n$ (where $n > 1$), each player plays one game with each of the other players and rules are such that no ties can occur. Let w_r and l_r be the number of games won and lost, respectively, by P_r. Show that $\sum_{r=1}^{n} w_r^2 = \sum_{r=1}^{n} l_r^2$.

16. A game of solitaire is played as follows. After each play, according to the outcome, the player receives either a or b points (a and b are positive integers with a greater than b), and his score accumulates from play to play. It has been noticed that there are 35 non-attainable scores and that one of these is 58. Find a and b.

17. Prove that it is impossible for seven distinct straight lines to be situated in the Euclidean plane so as to have at least six points where exactly three of these lines intersect and at least four points where exactly two of these lines intersect.

18. Let A be a set of 2*n* points in the plane, no three of which are collinear. Suppose that *n* of them are colored red and the remaining *n* blue. Prove or disprove: there are *n* closed line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.

19. (a) How can a farmer plant 7 trees in 6 rows of 3 each?
(b) How can a farmer plant 9 trees in 8 rows of 3 each?
(c) How can a farmer plant 9 trees in 9 rows of 3 each?
(d) For n trees in r rows of s each, what is the highest attainable value of the ratio rs/n ?

20. There are seven persons and seven committees. Each committee is to have three persons. Can you share the committees out to the people so that each person is on the same number of committees?

21. A (v, b, r, k, λ)-configuration on a set of v elements is a collection of b k-subsets such that

(i) each element appears in exactly r of the k-subsets,
(ii) each pair of elements appears in exactly λ of the k-subsets.

Prove that $k^r \geq v^\lambda$ and determine the value of b when equality holds.

22.
(ARML, 2002) At Archimedes Academy, the faculty is concerned about students' tendency to form cliques and it hires an anthropologist to study the cliques. The anthropologist finds that the cliques at AA satisfy the following three conditions:

A1. For any two students, there is exactly one clique of which they are both members.

A2. If a student a is not a member of a clique C, then there exists exactly one clique D of which a is a member and that has no members in common with C.

A3. There are three students that are not all members of the same clique.

Terminology note: A clique C is called *exclusive* of clique D if either $C = D$, or they have no members in common.

Also, S denotes the set of all students in the AA, and $\#C$ denotes the number of members in the clique C.

1. Determine a collection of cliques in AA if
   (i) $S = \{a, b, c\}$,
   (ii) $S = \{a, b, c, d\}$,
   (iii) $S = \{a, b, c, d, e, f, g, h, i\}$
2. Prove that if C is exclusive of D, and D is exclusive of E, then C is exclusive of E.
3. (i) Prove that if C and D are exclusive, then $\#C = \#D$.
   (ii) Prove that if C and D are not exclusive, then $\#C = \#D$.
   Some more notation: $[C]$ denotes the set of all cliques exclusive of C; and $\#[C]$ denotes the number of elements in this set.
4. Prove that $\#[C] = \#C$.
5. Prove that if C and D are distinct cliques, then $\#[C] = \#[D]$.
6. Prove that the number of students in the AA must be a perfect square.

Faculty at Hausdorff High were similarly concerned and called in the same anthropologist. At HH the anthropologist found that the cliques satisfy the following four conditions:

H1. For any two students, there is exactly one clique of which both are members.

H2. For any two cliques, there is exactly one student who is a member of both.

H3. There exist three students who are not all members of the same clique.

H4. Every clique has at least three students.

If the number of members in some clique *C* is *n*, find <u>with proof</u> the total number of students at HH in terms of *n*.

Definition: A *subplane* Π' of a projective plane Π is a projective plane whose points are a subset of the points of Π and in which each line is a subset of a line of Π.

23. Suppose that Π' is a subplane of a finite projective plane Π, and assume that $\Pi' \neq \Pi$. Let the orders of Π and Π' be n and m respectively. Prove that

$$n = m^2 \quad \text{or} \quad m^2 + m \leq n.$$

24. If Π' is a subplane of a finite projective plane Π and $n = m^2$ where *n* and *m* are the orders of Π and Π' respectively, prove that each line of Π contains at least one point of Π'.