Counting: Binomial Coefficients Everywhere!

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Problem 1. How many different ways are there to pick two of the five students sitting near the front to volunteer to come up to the board?

Problem 2. What two different meanings of the word "pick" might lead to two different answers to the previous problem? (I think the context implies the smaller of the two numbers is the one we want.)

Problem 3. Explain why $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. This immediately leads to Pascal's Triangle, which will come up in quite a different way later!

Problem 4. How many subsets are there of a set with 5 elements?

Problem 5. How many ways are there to choose two teams of two from among the five students sitting up front here?

Problem 6. Explain why $\binom{5}{1}\binom{4}{2} = \binom{5}{2}\binom{4}{1} = 3 \cdot \binom{5}{3}$. Hint: count the same process in three different ways.

Problem 7. Prove that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Is it easy enough to use the results of Problem 3, or is there a better way?

Problem 8. How many triangles can be formed in a 2 by n grid of dots? How many zigzags? (A zigzag is like a polygon: no edges crossing itself. But it doesn't have to end at the start.)

Problem 9. Explain why $n \cdot \binom{n}{n} + (n-1) \cdot \binom{n}{n-1} + \cdots + 2 \cdot \binom{n}{2} + 1 \cdot \binom{n}{1} = n \cdot 2^n$.

Problem 10. How many ways are there to choose five students to sit up front in this room? (What features of the five students do I care about? What if I don't care which students I get at all? What if I only care about male vs female? What if I only care what grade they're in?)

Problem 11. How many ways are there to seat 3 boys and 3 girls in a row of s seats, so that every boy is next to at least one girl, and every girl is next to at least one boy?

Problem 12. What if no girl may be next to a boy (and of course vice-versa)?

Problem 13. What if there are b boys and g girls?

Here's another application of binomial coefficients: the **binomial transform** of a sequence, or the **inverse binomial transform**. The binomial transform, given the first diagonal of the difference table, produces the sequence by multiplying each term by the appropriate binomial coefficients (which ones are appropriate, and why?). The inverse binomial transform, given the sequence, produces the first diagonal of the difference table. These are key ingredients in the method of **finite differences**, which is a very nice version of calculus where you deal with integers instead of real numbers (at least at first), and avoid thereby all the messiness of limits.

Problem 14. Given the sequence $a_n = 1, 4, 7, 10, 13, \ldots$, I hope you can all see that the differences $Da_n = 3, 3, 3, 3, \ldots.$ That helps you calculate the next term, so do that, and then also use it to find a general formula for a_n in terms of n.

Problem 15. If the first term of a sequence is 7, and it has first difference a constant 2, write a formula for the sequence.

Problem 16. How about a formula for the sequence 2, 9, 22, 41, 66, 97, 134, …?

Problem 17. If the first term of a sequence is 11, the first term of its first differences is 5, and the second difference is a constant 12, write a formula for the sequence.

Problem 18. What about 0, 6, 24, 60, 120, 210, 336, …? This one might be amenable to a good guess-and-check, too, as well as the finite differences method.

Problem 19. Now let's work the other way: if $a_n = mx + b$, what will Da_n be equal to? What if a_n has a quadratic pattern? Cubic?

Problem 20. If the first term of a sequence is w, the first term of its first differences is x, the first term of its second differences is y, and the third difference is a constant z, write a formula for the sequence.

Problem 21. Instead of powers of x like x^2 and x^3, why is it better to write our polynomials in terms of binomial coefficients?

Problem 22. What does all the above stuff about differences tell you about the sum of a sequence?

Problem 23. Make a nice formula for 1, 2, 4, 8, 16, 31, 57, 99, 163, 256, 386, 562, 794, 1093, 1471, 1941, 2517, 3214, 4048, …

Problem 24. What happens when you try this same technique on 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, …? Compare this to Problem 8.

Problem 25. What about if you try it on 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …?