Non-Euclidean III.36

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1. INTRODUCTION. Among the theorems of plane geometry, a privileged position is held by those that are true in neutral geometry, that is, without either assuming or denying the parallel postulate. Such, for example, are the first twenty-eight propositions of Euclid's *Elements*, including the triangle congruences SAS (I.4),[1] SSS (I.8), ASA (I.26), and the base angles of an isosceles triangle are equal (I.5). Some concurrence theorems, such as the three angle bisectors of a triangle meeting in a point, are easily proved in neutral geometry (cf. (IV.4)). Others, such as the one pertaining to the three altitudes of a triangle (if two of them meet, then the third also meets in the same point) are true in neutral geometry, but more difficult to prove [**4**, pp. 369, 387, 399, 430].

On the other hand, the Pythagorean theorem (I.47), proved either by using Euclid's theory of area or by using similar triangles, is irrevocably tied to the Euclidean parallel postulate. There is an analogue in hyperbolic geometry (discovered by Bolyai and Lobachevskii) that says $\sin\overline{c} = \sin\overline{a}\,\sin\overline{b}$, where $\overline{a}, \overline{b}, \overline{c}$ are the angles of parallelism associated with the sides a and b and the hypotenuse c of a right triangle [**4**, p. 406]. There is also a non-Euclidean version using area [**10**], but its statement is not entirely satisfactory, because the figure on the hypotenuse is made to depend on the figures on the two sides.[2]

The purpose of this article is to present a uniform formulation of III.36 that is valid in neutral geometry, including the case of Euclidean, hyperbolic, elliptic, and spherical geometries. Since it is easy to show that III.36 implies I.47 in Euclidean geometry (see Theorem 4), perhaps one should regard III.36 as an even more basic theorem about area than the famous Pythagorean theorem.

2. EUCLIDEAN III.36. Proposition 36 of Book III of Euclid's *Elements* [**2**] is the statement that if P is a point outside a circle, if PA is a tangent to the circle, and if PBC is a secant line, then $PA^2 = PB \cdot PC$ (Figure 1).

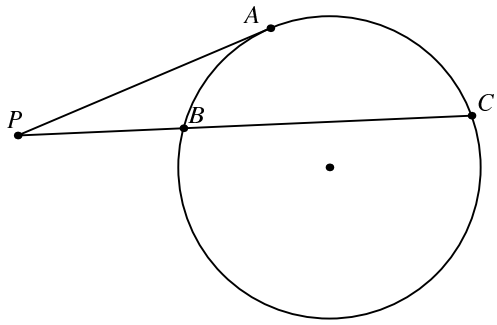

Figure 1.

[1]References of the form I.4 or III.36 are to the corresponding Book and Proposition of Euclid's *Elements* [**2**].

[2]Piel's theorem [**10**] says if A is a right-angled rhombus on the side of a right triangle, whose other angle is α, and B is a right-angled rhombus on side b, with other angle β, then there is a rhombus C on the hypotenuse c of the triangle, with angles α and β, and $A + B = C$ in the sense of area.

There are two ways to understand this statement. In Book III, Euclid takes PA^2 to mean the square on the side PA, and $PB \cdot PC$ to mean the rectangle with sides PB and PC. The equality of the two is in the sense of area.

Euclid never defines what he means by equality of area, but by reading between the lines in Book I, we can infer that what he means is this: two plane figures P, Q are *equidecomposable* if one can cut P into a finite number of triangles, and then reassemble them to form Q. Two figures P, Q are *equal* (in the sense of area) if there is a third figure R such that $P + R$ and $Q + R$ ("+" signifying a nonoverlapping union) are equidecomposable. Hilbert showed in his *Foundations of Geometry* [**5**] that this gives an equivalence relation on plane figures, with all the properties one would expect of a notion of area, and that this theory works in an arbitrary *Hilbert plane*, i.e., assuming only Hilbert's axioms of incidence, betweenness, and congruence, without the parallel postulate or axioms of continuity [**4**, sec. 22].

Euclid's proof of III.36 makes use of the Pythagorean theorem (I.47) in the sense of area, and the results of "geometric algebra" in Book II.

The other way to understand III.36 is by working in the Cartesian plane over a field F. Then we can take PA, PB, and PC to mean the lengths of the corresponding segments, as elements of the field F, and $PA^2 = PB \cdot PC$ becomes a statement of arithmetic in the field F. This form of III.36 is easy to prove using similar triangles [**4**, 20.9].

The two interpretations of III.36 are related by the usual measure of area function in Euclidean geometry, which to each figure P assigns an element $\alpha(P)$ in the ground field F, called its area [**4**, 23.2]. One can show that two figures P, Q are equal in the sense of area if and only if $\alpha(P) = \alpha(Q)$ [**4**, 23.7]. Since the area of a rectangle is the product of the lengths of its sides, the two statements are equivalent.

3. NEUTRAL III.36. There are no rectangles in non-Euclidean geometry, so we must change the formulation of III.36 slightly for it to make sense in neutral geometry. Define a *semi-rectangle* to be a quadrilateral with opposite sides equal, and at least one right angle. It follows that the angle opposite the right angle is also right, and that the two remaining angles are equal (Figure 2). If all four sides are equal, we call it a *semi-square*. Note that for any given segments a and b there is a unique semi-rectangle with sides a and b, obtained by gluing two right triangles with legs a and b together along their hypotenuses.

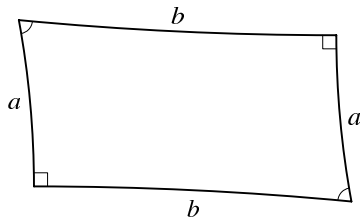

Figure 2.

Theorem 1. *Let P be a point outside a circle, let PA be a tangent line, and let PBC be a secant line. Then the semi-square on PA is equal (in the sense of area) to the semi-rectangle on PB and PC (Figure 3).*

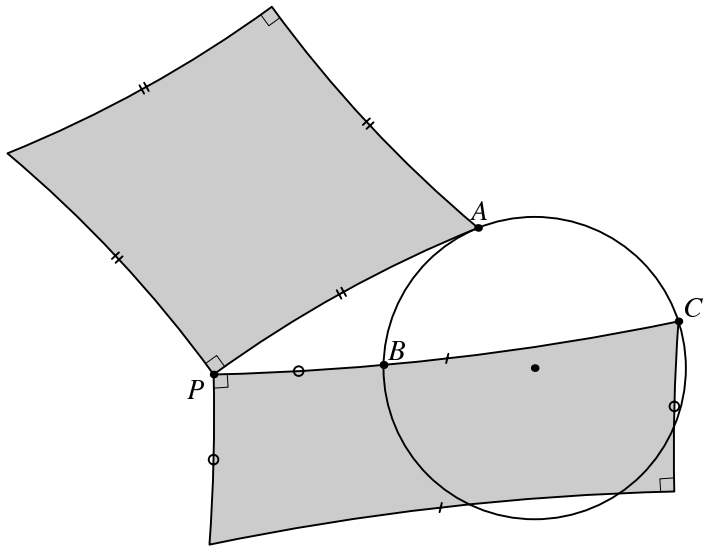

Figure 3.

Proof. We will show that this theorem is true in any Hilbert plane with (E) (this is the circle-circle intersection axiom, which states that if one circle has at least one point inside and at least one point outside a second circle, then the two circles intersect [**4**, p. 108]), and also in elliptic and spherical geometry.

Case 1. In a Hilbert plane with (P), the parallel axiom, the semi-square and semi-rectangle become an ordinary square and rectangle, and the proofs of Euclidean III.36 mentioned earlier are valid [**4**, 12.4; 20.9].

Case 2. Consider the Poincaré disk model of hyperbolic geometry over an ordered Euclidean field F, using a defining circle Γ of radius 1 centered at the origin [**4**, sec. 39].

Lemma 2. *In the Poincaré model, consider a Poincaré right triangle OAB, where $O = (0, 0)$ is the origin, and $A = (x, 0)$ and $B = (0, y)$ are points on the x- and y-axes. If the angles at A and B are α and β, respectively, let $\delta = \pi/2 - \alpha - \beta$ be the "defect" of the triangle. Then* $\tan(\delta/2) = xy$.

Proof of Lemma 2. (See Figure 4.) The Poincaré line AB is part of a circle Δ orthogonal to Γ. Let its center be C. Join CA, CB, and drop perpendiculars CD, CE to the two axes. Then $\angle ACD = \alpha$ and $\angle BCE = \beta$, so $\angle ACB = \delta$. Let A' be the other intersection of Δ with the x-axis. Since Δ is orthogonal to Γ, A' is the circular inverse of A [**4**, 37.3], namely, $A' = (1/x, 0)$. Since the arc AB subtends an angle δ at C, it subtends the angle $\delta/2$ at A' (III.20). Thus $\tan(\delta/2) = OB/OA' = y/(1/x) = xy$.[3] ■

[3]If we translate this result into a formula intrinsic to hyperbolic geometry, using the methods of [**4**, secs. 39,42], then we obtain the formula $\tan(\delta/2) = \cos F(a/2) \cos F(b/2)$ given by Lobachevskii in his first publication on non-Euclidean geometry [**8**, p. 36]. Here δ is the area of the right triangle with legs a, b, and F, in Lobachevskii's notation, denotes the angle of parallelism of a given segment. In fact it was this formula, which I saw for the first time recently in Lobachevskii's work, and its similarity to a non-Euclidean analogue of III.36 I had discovered earlier [**4**, Exercise 42.21], that led to the discovery of the main theorem of this paper.

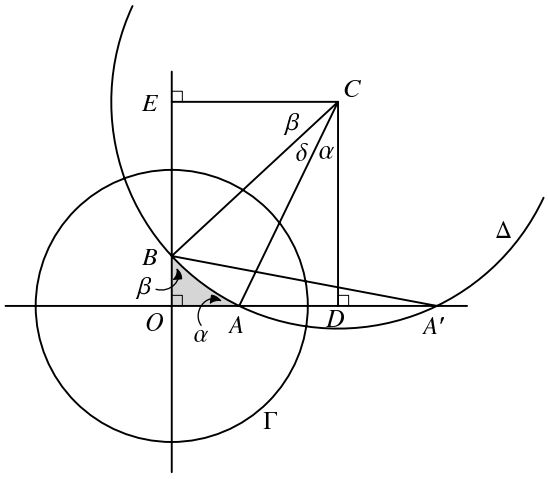

Figure 4.

To prove our theorem in the Poincaré model, suppose given P, A, B, C as in the theorem. Use a rigid motion of the Poincaré model to move P to the center of Γ [**4**, 39.5]. Then the lines PA and PBC become Euclidean lines through P, and the circle is transformed into a Euclidean circle [**4**, 39.8]. Applying the Euclidean III.36 in the ambient plane, we find $PA^2 = PB \cdot PC$, where PA, PB, PC denote Euclidean lengths.

Now by Lemma 2, PA^2 is equal to $\tan(\delta/2)$, where δ is the defect of the hyperbolic isosceles right triangle with both legs equal to PA. On the other hand, $PB \cdot PC$ is equal to $\tan(\delta'/2)$, where δ' is the defect of the hyperbolic right triangle with legs PB, PC. Thus $\tan(\delta/2) = \tan(\delta'/2)$, implying that $\delta = \delta'$.

Now in hyperbolic geometry the defect of a triangle gives a measure of area function [**4**, 36.2], so we conclude that these triangles are equal in the sense of area [**4**, 36.6]. By gluing two copies of each of these triangles together along their hypotenuses, we find the semi-square on PA is equal (in the sense of area) to the semi-rectangle with sides PB, PC.

Case 3. To show that our theorem holds in any Hilbert plane with (E), we use the division of all Hilbert planes into three types [**4**, 34.7]: a Hilbert plane is *semi-hyperbolic* if the sum of the angles of every triangle is less than two right angles; *semi-Euclidean* if the sum of the angles of every triangle is equal to two right angles; and *semi-elliptic* if the sum of the angles of every triangle is greater than two right angles. We also use the classification theorem of Pejas [**4**, 43.7], which says that any semi-hyperbolic Hilbert plane satisfying (E) (respectively any semi-Euclidean plane, or any semi-elliptic plane satisfying (E)) is isomorphic to a full subplane (definition [**4**, p. 423]) of the Poincaré model over a Euclidean field F (respectively the Cartesian plane over F, or a certain non-Archimedean semi-elliptic plane described in [**4**, Exercise 34.14b]).

So if our plane is semi-Euclidean, it is a subplane of the Cartesian plane over a field F. Our figure therefore lies also in the Cartesian plane, and the result follows from the Euclidean case. If our plane is semi-hyperbolic, it is a subplane of the Poincaré model over a field F, and the result follows from Case 2.

It remains to treat the semi-elliptic case. This one is a subplane of a particular model constructed out of spherical geometry. Indeed we will show more generally that our theorem holds in elliptic and spherical geometry. (These cases were not mentioned in

Pejas's theorem, because they do not satisfy the betweenness axioms, hence do not fall within the general notion of Hilbert planes.)

Case 4. The theorem holds in spherical and elliptic geometry. By spherical geometry, we mean geometry on the surface of a sphere, where the great circles are taken as lines. This is sometimes called double elliptic geometry. The (single) elliptic geometry is obtained by identifying antipodal points on the sphere, so that two lines will intersect in only one point instead of two. For simplicity, we will assume that our figure is entirely contained in the southern hemisphere, with P at the south pole. Then we can treat both cases at once.

We take our sphere to be of radius $1/2$, and set it with its south pole on the origin of a Cartesian plane. Then we use the stereographic projection from the north pole to project the sphere onto the plane. The southern hemisphere is mapped under this correspondence to the interior of the unit circle in the plane.

Thus we get a model of the spherical geometry on the plane, where lines become circles containing two diametrically opposite points of the unit circle, and the elliptic geometry becomes the interior of the unit circle plus the opposite points of the unit circle identified. This representation is conformal, and sends (spherical) circles to plane circles (see [**1**, pp. 171–174] or [**7**, pp. 171–172]).

Lemma 3. *In the plane model of spherical geometry, let OAB be a right triangle with right angle at the origin $O = (0, 0)$, and with $A = (x, 0)$ and $B = (0, y)$. Let $\delta = \alpha + \beta - \pi/2$ be the "excess" of the triangle. Then* $\tan(\delta/2) = xy$.

Proof of Lemma 3. (See Figure 5.) This is completely analogous to the proof of Lemma 2. The spherical line AB is part of a circle Δ with center C meeting the unit circle Γ in two diametrically opposite points D and E. One sees as before that δ is equal to the angle $\angle ACB$. Let A' be the other intersection of the x-axis with Δ. Then by (III.35) we have $OD \cdot OE = OA \cdot OA'$, so $A' = (-1/x, 0)$. The angle at A' subtended by the arc AB is $\delta/2$, giving $\tan(\delta/2) = OB/OA' = xy$.[4] ■

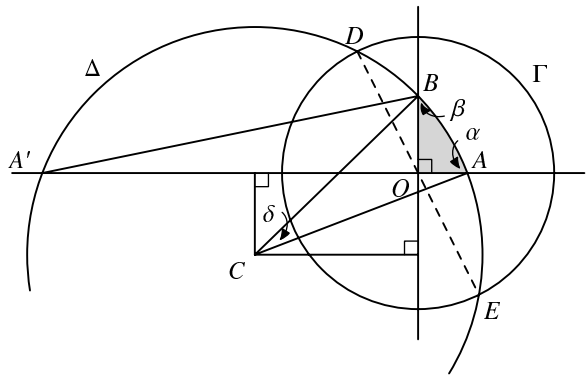

Figure 5.

[4]If the segments OA, OB are projections of spherical segments subtending angles of a, b at the center of the sphere, then these same segments subtend angles of $a/2, b/2$ at the north pole, and we obtain the usual formula $\tan(\delta/2) = \tan(a/2)\tan(b/2)$ for the excess of a spherical right triangle [**6**, Exercise 8, p. 284].

To prove our theorem on the sphere, move the figure by a rotation so that P is at the south pole, then project onto the plane. Let A', B', C' be the projections of the spherical points A, B, C. Then apply Euclidean III.36 to the plane figure and deduce that $PA'^2 = PB' \cdot PC'$. According to Lemma 3, PA'^2 is equal to $\tan(\delta/2)$, where δ is the excess of the spherical isosceles right triangle with both legs PA (which is projected conformally onto a plane triangle with both legs PA'). Similarly, $PB' \cdot PC' = \tan(\delta'/2)$, where δ' is the excess of the spherical right triangle with legs PB, PC (as projected in the plane). We infer that $\delta = \delta'$. Now on the sphere the excess gives a measure of area function, and we conclude that the triangles are equal in the sense of area.[5] Gluing together two copies of each, we find the semi-square with side PA is equal to the semi-rectangle with sides PB, PC.[6] ■

4. A NEW(?) PROOF OF THE PYTHAGOREAN THEOREM.[7]

Theorem 4. *In a semi-Euclidean Hilbert plane satisfying* (E)*, the square on the hypotenuse of a right triangle is equal (in the sense of area) to the sum of the squares on the two sides.*

Proof. Let ABC be a triangle with a right angle at C. Let Γ be a circle with diameter AC, and let Δ be a circle with diameter BC. Using the semi-Euclidean hypothesis, Γ and Δ will meet at a point D on the hypotenuse AB (Figure 6).[8]

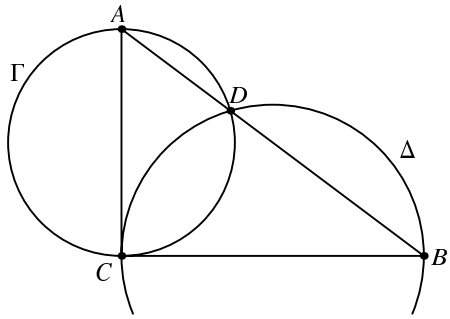

Figure 6.

[5]To conclude equality in the sense of area, we need the spherical form of the Bolyai–Gerwien theorem [**3**]. See also [**7**, pp. 157–158].

[6]While the statement of the theorem is pleasing in its generality, the proof given here, depending as it does on the classification of Hilbert planes and calculations in particular models that can be reduced to the Euclidean case, is not so satisfying. One would hope for a uniform proof, valid in neutral geometry, that did not depend on special cases.

[7]E. Loomis [**9**] has collected more than 360 proofs of the Pythagorean theorem, and I suppose many more have been discovered since the publication of his book. Therefore I hesitate to claim this is a new proof, and prefer to call it a new(?) proof. The diagram I use appears in Loomis's proof number 88 [**9**, p. 80], but he makes a reasoning with similar triangles, and concludes the algebraic form of the theorem $a^2 + b^2 = c^2$. Since my proof deals in areas only, it is different. By the way, Loomis refers to the algebraic formulation of III.36 in several books, such as Davies's *Legendre* (1858), and Wentworth's *Geometry* (1895), but unaccountably does not refer to Euclid's own III.36.

[8]Let D be the intersection of the two circles. From our semi-Euclidean hypothesis, it follows that the angle $\angle ADC$ is right (III.31) and similarly $\angle BDC$ is right. Thus AD and BD are in a straight line, i.e., D lies on AB. This point fails in hyperbolic geometry, so one cannot get a hyperbolic analogue of (I.47) by this method.

Applying Theorem 1 to the point B outside the circle Γ, we find the square on BC is equal to the rectangle $BD \times BA$. Applying Theorem 1 to the point A and the circle Δ, we find the square on AC is equal to the rectangle $AD \times AB$. Adding, the sum of the squares on AC, BC is equal to the sum of the two rectangles, which is just the square on AB.

5. EXTENSIONS. An immediate corollary of Theorem 1 is the following result for two chords to a circle.

Theorem 5. *Let P be a point outside a circle, and let PBC, PDE be two chords. Then the semi-rectangle on PB and PC is equal (in the sense of area) to the semi-rectangle on PD and PE.*

Also, by the same method of proof as for Theorem 1, we can obtain a neutral geometry version of Euclid's III.35:

Theorem 6. *Let two chords AB, CD of a circle meet at a point P interior to the circle. Then the semi-rectangle on PA, PB is equal (in the sense of area) to the semi-rectangle on PC, PD.*

These theorems allow us to give a uniform definition of the "power" of a point with respect to a circle in neutral geometry. The usual definition of the power of a point P with respect to a circle in Euclidean geometry is to take a line through P meeting the circle at A and B. Then the *power* of P is the number obtained by multiplying the signed distances PA and PB [**6**, p. 232]. Because of III.35 and III.36, this is independent of the chord chosen.

In our case the power of a point will not be a number, but an area, using the notion of equivalence in the sense of area mentioned earlier. (To be precise, it is an element of the ordered abelian group G whose positive elements are equivalence classes of plane figures in the sense of area, and where addition corresponds to nonoverlapping union.)

Definition.[9] If P is a point outside the circle Γ, the *power* of P with respect to Γ is the area of the semi-rectangle on PB and PC, where PBC is any chord. If P is inside Γ, the *power* of P is the negative of the area of the semi-rectangle on PA and PB, for any chord AB passing through P. (This is well-defined because of Theorems 1 and 6.)

Proposition 7. *If two circles meet in two points A and B, then the line AB is equal to the locus of points P for which the power is the same for both circles.*

This is easy to see from the definition. Using this result, we obtain a proof, in neutral geometry, for the following theorem (see the cover illustration for [**4**]):

Theorem 8. *Let three circles meet in pairs. If two of the common chords of the three circles, taken two at a time, meet in a point P, then the third common chord also passes through P.*

[9] Sommerville [**11**, p. 212] shows in elliptic geometry that if PBC is a chord to a circle, then the quantity $\tan(PB/2)\tan(PC/2)$ depends only on P. He calls this quantity the power of the point. The analogous proof, using hyperbolic trigonometry, shows that the quantity $\cos F(PB/2)\cos F(PC/2)$ is independent of the chord through P, so it could be called the power of the point in that case. The advantage of our method is to give a uniform definition valid in all cases.

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