Rigid motions, symmetry and crystals.

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Rigid motions on a plane.

A *rigid motion* is a map of a plane to itself which preserves distances and angles.

1. Show that a parallel translation, a central symmetry, a rotation and a reflection are rigid motions.

2. Show that a composition of two rigid motions is a rigid motion.

3. Show that a composition of two central symmetries is a parallel translation.

4. Show that a composition of two reflections is a parallel translation or a rotation.

5. What is a composition of two rotations with different centers?

6. Let ABC and $A'B'C'$ be congruent triangles. Show that there is exactly one rigid motion which send A to A', B to B' and C to C'.

7. Show that any rigid motion is a translation, a rotation, a reflection or a composition of a reflection and a translation along the line of reflection.

Rigid motions in geometric problems.

Rigid motions and symmetry are very useful in some geometric problems. Here are some examples.

1. Given an angle $\angle ABC$ and a point D inside this angle. Construct a segment with the endpoints on the sides of the angle $\angle ABC$ and the midpoint D.

2. There is a regular polygon with 10 vertices. Two players play the following game. Each player in his turn draws a diagonal which does not cut previously drawn diagonals. The player who can not make a move looses. Who can always win in this game?

3. Given a line l and points A and B on the same side of l. Find the point X on l such that $AX + XB$ is the smallest.

4. Given an acute triangle ABC. Find points P, Q and R, one on every side of this triangle, such that the triangle PQR has the smallest perimeter.

5. On the sides of a triangle ABC the squares $ABMN$ and $BCPQ$ are constructed (outside of the triangle). Show that the centers of the squares and the midpoints of AC and QM are vertices of another square.

6. On the sides of a triangle the equilateral triangles are constructed (outside of the triangle). Show that the centers of these equilateral triangles form another equilateral triangle.

7. Find a point X inside the triangle ABC such that $AX + BX + CX$ is the smallest.

Rigid motions in space.

One can define rigid motion in space in the same way as on a plane.

1. A parallel translation, a reflection in a plane and a rotation about some axis are rigid motions.

2. Let F be a rigid motion which does not move a point P. Then F is either a rotation or a reflection composed with some rotation (may be on 0 degrees).

3. Any rigid motion in space is a rotation, a parallel translation, a reflection or a composition of these motions.

Groups of symmetries.

Let G be some set of rigid motions (on a plane or in space) satisfying two properties

(1) if a rigid motion is in G then its inverse is also in G;
(2) a composition of two motions from G is again in G.

We call such G a *group.* A typical example of a group is the set of all rigid motions preserving a given geometric figure. A group of symmetries of a rectangle contains two reflections, a central symmetry and the identity map. So it has four elements.

1. If the group of symmetries of a plane figure contains more than one central symmetry, then it has infinitely many central symmetries.

2. Show that a polygon has at most one center of symmetry.

3. Given a hexagon such that any two opposite sides are parallel and congruent. Show that this hexagon is centrally symmetric.

4. Describe all quadrilaterals with group of symmetries of 4 elements.

5. Find the number of symmetries of a parallelogram, a square, an equilateral triangle, a regular tetrahedron, a cube and a dodecahedron.

6. Show that the group of symmetries of a cube contains a group of symmetries of a tetrahedron. (Hint: inscribe a regular tetrahedron in a cube.)

7. List the angles of rotations for all rotations which are symmetries of a dodecahedron?

8. Show that any group of rigid motions with finitely many elements fixes a point.

9. Show that any finite group of rigid motions on plane is the group of symmetries of a regular polygon or of some quadrilateral, or just a group of rotations on the multiples of the angle $\frac{360^\circ}{n}$.

Crystals and crystallographic groups.

A set M of points on a plane (in space) is called *regular if*

(1) every circle (ball) contains finitely many points from M;
(2) every circle (ball) with sufficiently large radius contains a point of M;
(3) for any two points x and y from M there is a rigid motion from the group of symmetries of M which maps x to y.

It is clear that a regular set M has a large (infinite) group of symmetries G. A group G of rigid motions is called a *crystallographic group* if by applying all motions from G to some point x one gets a regular set M.

1. Let G be a crystallographic group on a plane (in space) and T be the set of all translations from G. Show that T is a group.

2. Show that T can be obtained starting with two (three) parallel translations on a plane (in space) by applying compositions and taking inverse.

3. Let x be a point of a regular set M and G_x be the set of symmetries from G preserving x. Show that G_x is a finite group.

4. List all G_x possible for crystallographic groups G on a plane. You should get 10 different groups.

5. Describe all crystallographic groups on a plane. You should get 17 different groups.

6. Think about problems 4 and 5 in space. There are 32 possible G_x and 230 different crystallographic groups! (Actually in crystallography all these groups have names. If you take exam in crystallography, you have to know them all.)

7. Let G be a crystallographic group. There exists a polygon (a polyhedron for the space) such that its images under the symmetries from G cover the whole plane (space) without overlapping.

Place several atoms inside this polyhedron, proliferate them along the space. You get a picture similar to how atoms are placed in real crystals.