Is it possible to cut a cube into pieces and to assemble a tetrahedron? Hilbert's problem and Dehn's theorem

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1 Hilbert's Problem Three.

Is it possible to cut a cube by finitely many planes and assemble, out of the polyhedral pieces obtained, a regular tetrahedron of the same volume?

This is a slight modification of one of the 23 problems presented by David Hilbert in his famous talk at the Congress of Mathematicians in Paris, on August 8, 1900; it goes under the number 3. Hilbert's problems had a tremendous impact on Mathematics. Most of them were solved during XX century, and each has a very special history. Still, Problem Three stands exceptional in many respects.

First, this was the first of Hilbert's problems to be solved. The solution belonged to a 23 years old German geometer, Hilbert's student Max Dehn [3]. His article appeared two years after the Paris Congress, but the solution existed earlier, maybe, even before Hilbert stated the problem.

Dehn's proof (more or less the same as the one presented below) was short and clear, and it became one of the favorite subjects for popular lectures, articles, and books in geometry, like the one you are holding in your hands (I can recommend the book by Boltianskii [1] for a more extended exposition). But among working mathematicians, it was almost forgotten.

Certainly, the name of Dehn was not forgotten. He became one of the few top experts in topology of three-dimensional manifolds, and his work of 1902 has been never regarded as his main achievement; it is not even mentioned in Dehn's biography available on the web.

In 1976, American Mathematical Society published a two-volume collection of articles under the title "Mathematical Developments Arising from

Hilbert Problems" [5]. It was a very solid account of the three quarters of century of history of the problems: solutions, full and partial, generalizations, similar problems, and so on. This edition contains a thorough analysis of 22 of 23 Hilbert's problems. And only Problem Three is not discussed there. The opinion of the editors is obvious: no developments, no influence on Mathematics; nothing to discuss.

How strange it seemed just a couple of years later! Dehn's theorem, Dehn's theory, Dehn's invariant became one of the hottest subjects in geometry. This was stimulated by then new-born K-theory, an exciting domain developed along the borderline between algebra and topology. We will not follow this development, but will just revise the theorem and its proof.

2 For a similar problem in the plane the answer is yes.

Theorem 2.1 *Let P_1, P_2 be two polygonal domains in the plane having the same area. Then it is possible to cut P_1 into pieces by straight lines and to assemble Of these pieces P_2.*

Proof. First, it is clear that it is sufficient to consider the case when P_2 is a rectangle with the sides 1 and area P_1; in doing this, we can shorten the notation of P_1 to just P.

Second, since any polygonal domain can be cut into triangles, we can reduce the general case to that of a triangle (see Figure 1).

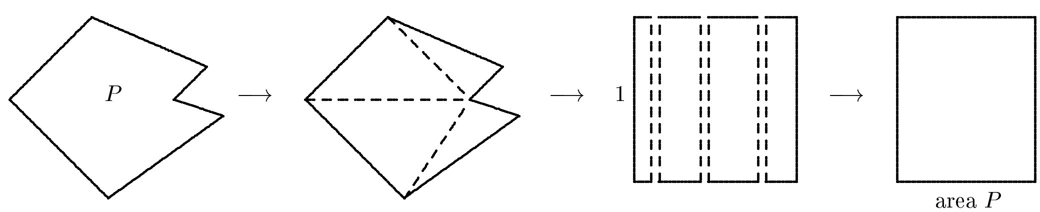

Figure 1:

Third, we need to remake, by cutting and pasting, a triangle into a rectangle with one if the sides having length one. This is done, in four steps, on Figure 2.

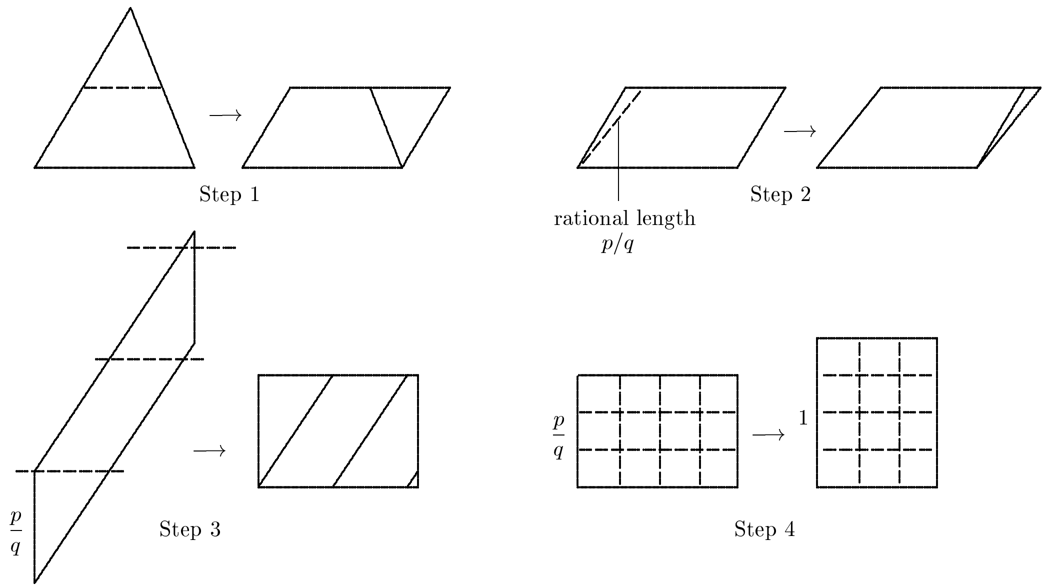

Figure 2:

We do this in four steps. First, we make a parallelogram out of our triangle (Step 1). Then we cut a small triangle on one side of the parallelogram and attach it to the other side in such a way that the length of one of the sides of the parallelogram becomes rational, p/q (Step 2). On Step 3, we make this parallelogram a rectangle (the number of horizontal cuts needed depends on the shape of the parallelogram). On the final step, we cut the rectangle into pq equal pieces by $p-1$ horizontal lines and $q-1$ vertical lines (with the understanding that it is the vertical side of the rectangle that has the length p/q); then we rearrange these pq pieces into a rectangle with the length of the vertical side being 1.

3 A planar problem which does not look similar to Hilbert's Problem Three, but has a similar solution.

Is it possible to cut a 1×2 rectangle into finitely many smaller rectangles with sides parallel to the sides of the given rectangle and to assemble a $\sqrt{2} \times \sqrt{2}$ square?

The answer is *NO*. The proof is more algebraic than geometric, but still, unlike the Hilbert Problem, it requires a small geometric preparation.

3.1 A geometric preparation.

Let us given two rectangles with vertical and horizontal sides (below, we will call such rectangles briefly VH-rectangles), and suppose that is is possible to cut them into smaller VH-rectangles such then the pieces of the first are equal (congruent) to the pieces of the second.

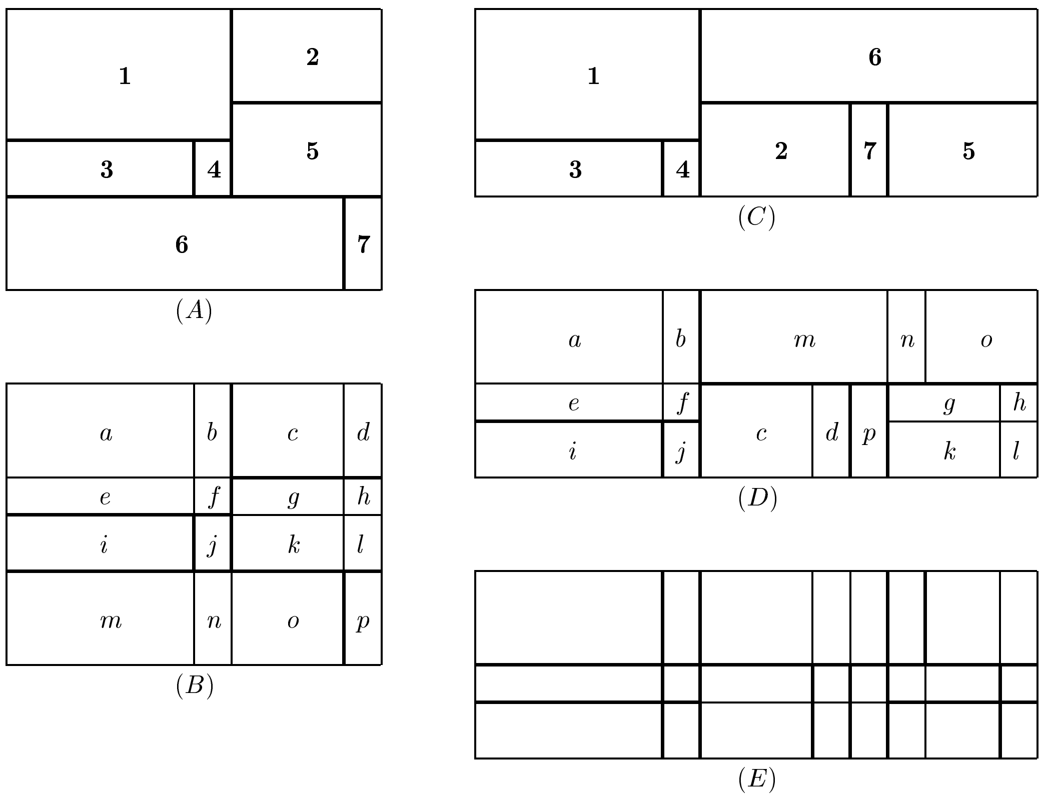

Figure 3:

Then there exists a collection of N (still smaller) VH-rectangles such that each of the given rectangles can be obtained by a sequence of $N-1$ *admissible moves*. An admissible move: we take two of our small rectangles having equal widths or equal heights and attach them to each other vertically or horizontally, creating one rectangle of the same width or height. Thus our process of cutting is replaced by a more strict process of attaching of rectangles. How to do this, is shown on Figure 3.

Suppose that two rectangles are cut into equal pieces as requested by Problem (rectangles (A) and (C) on Figure 4; equal pieces are marked there by the same Arabic numbers). Then we extend the sides of the pieces to the whole width or length of the rectangle (see rectangle (B) of Figure 4). Some of the pieces of division are cut into smaller pieces (marked by Roman letters in rectangle (B): so **1** becomes a union of a, b, e and f, **2** becomes a union of c and d, etc.) Then we divide in the same way the pieces of the second given rectangle (see rectangle (D) of Figure 4; we break the rectangle **1** of rectangle (C) into pieces congruent to a, b, e, f, the rectangle **2** into pieces c, d, and so on). We obtain a new division of the second given rectangle into smaller rectangles, and again extend the sides of these smaller pieces to the whole width or length of the rectangle (see rectangle (E) of Figure 4). These last pieces form our collection. Obviously we can assemble the second rectangle, (C), from these pieces using the admissible moves. Other admissible moves produce, out of our small rectangles, the parts of the finer division of the rectangle (A) (that is, $a, b, c, \ldots, o, p$), and out of this part we can assemble, using admissible moves, the rectangle (A). The geometric preparation is over.

3.2 An algebraic proof.

Let us have a finite collection of VH-rectangles of the total area 2. Then no more than one of the following two is possible:

– to compose out of these rectangles a 1×2 rectangle using only admissible moves;

– to compose out of these rectangles a $\sqrt{2} \times \sqrt{2}$-square using only admissible moves.

This is what we need to answer negatively the question of these section.

Let $w_1, \ldots, w_N$ be the widths of the rectangles of our collection (N being the number of these rectangles), and $h_1, \ldots, h_N$ be their heights.

Consider the sequence

$$1, \sqrt{2}, w_1, \ldots, w_N; \tag{1}$$

remove a member of this sequence if it is a linear combination, with rational coefficients, of the preceding members. (Thus, we do not remove 1; we do not remove $\sqrt{2}$, since it is irrational; we remove w_1, if and only if $w_1 = r_1 + r_2\sqrt{2}$, with rational r_1, r_2, and so on.) Let $a_1, \ldots, a_m$ be the remaining numbers (thus, $a_1 = 1, a_2 = \sqrt{2}$). It is important that each of the numbers (1) can be presented as a rational linear combination of the numbers $a_1, \ldots, a_m$ in a unique way[1].

Now, do the same with the sequence

$$1, \sqrt{2}, h_1, \ldots, h_N. \tag{2}$$

We will get the numbers $b_1, \ldots, b_n$ with $b_1 = 1, b_2 = \sqrt{2}$ such that each of the numbers (2) can be presented as a rational linear combination of the numbers $b_1, \ldots, b_n$ in a unique way.

Call a rectangle admissible, if its width is a rational linear combination of $a_1, \ldots, a_m$ and its height is a rational linear combination of $b_1, \ldots, b_n$. Let P be an admissible rectangle of the width w and the length h, and let $w = \sum_{i=1}^{m} r_i a_i$ and $h = \sum_{j=1}^{n} s_j b_j$ with rational r_i's an d s_j's. We define the *symbol* Symb(P) of the rectangle P as the rational $m \times n$ matrix $\|S_{ij}\|$ with $S_{ij} = r_i s_j$. We usually will use for the symbols the notation $\mathrm{Symb}(P) = \sum_{i,j} r_i s_j \, a_i \otimes b_j$ (which is simply the alternative notation for the matrix above). Thus, we regard the symbols as "formal rational linear combination" of the "expressions" $a_i \otimes b_j$. Such formal linear combinations cab be added in the obvious way; we consider two formal rational linear combinations $\sum_{i,j} t'_{ij} a_i \otimes b_j$, $\sum_{i,j} t''_{ij} a_i \otimes b_j$ equal if $t'_{ij} = t''_{ij}$ for all i, j.

Let P' and P'' be two admissible rectangles of equal heights or equal widths. Then we can merge these two rectangles into one rectangle, P, using an admissible move (see above). Obviously, P is also an admissible rectangle, and $\mathrm{Symb}(P) = \mathrm{Symb}(P') + \mathrm{Symb}(P'')$. Indeed, if P' and P'' have widths

[1] This is a standard theorem from linear algebra, but for the sake of completeness, let us give a proof. $1 = a_1$ is a rational linear combination of $a_1, \ldots, a_m$, so is $\sqrt{2} = a_2$. Assume, by induction, that all the numbers (1) preceding w_k are rational linear combinations of $a_1, \ldots, a_m$. If w_k is not a rational linear combination of preceding numbers, then it is one of a_j's, and hence is a rational linear combination of $a_1, \ldots, a_m$; if w_k is a rational linear combination of preceding numbers, then it is a rational linear combination of $a_1, \ldots, a_m$, since all the preceding numbers are rational linear combinations of $a_1, \ldots, a_m$. It remains to prove uniqueness. If two different rational linear combinations of $a_1, \ldots, a_m$ are equal, $\sum_{i=1}^{m} r'_i a_i = \sum_{j=1}^{m} r''_j a_j$, and s is the largest of $1, \ldots, m$, for which $r'_s \neq r''_s$, then $a_s = \sum_{i=1}^{s-1} \frac{r'_i - r''_i}{r''_s - r'_s} a_i$ which shows that a_s is a rational linear combination of preceding a_j's, in contradiction to the choice of $a_1, \ldots, a_m$.

$w' = \sum_{i=1}^m r'_i a_i$ and $w'' = \sum_{i=1}^m r''_i a_i$ and the same height $h = \sum_{j=1}^n s_j b_j$, then P has the width $w' + w'' = \sum_{i=1}^m (r'_i + r''_i) a_i$ and the height h, and

$$\begin{aligned}\mathrm{Symb}(P) &= \sum_{i,j} (r'_i + r''_i) s_j \ a_i \otimes b_j \\ &= \sum_{i,j} r'_i s_j \ a_i \otimes b_j + \sum_{i,j} r''_i s_j \ a_i \otimes b_j \\ &= \mathrm{Symb}(P') + \mathrm{Symb}(P'').\end{aligned}$$

Thus, if we have a collection of admissible rectangles, $P_1, \ldots, P_N$, and can assemble out of them, by $N-1$ admissible moves, a rectangle P, then $\mathrm{Symb}(P) = \sum_{i=1}^N \mathrm{Symb})P_i)$. If we can assemble in this way two different rectangles, P and P', then $\mathrm{Symb}(P') = \mathrm{Symb}(P)$. This proves our theorem, since the symbol of a 1×2 rectangle is $2(a_1 \otimes b_1)$, and the symbol of a $\sqrt{2}\times\sqrt{2}$ square is $a_2 \otimes b_2$ which is different.

4 Proof of Dehn's Theorem.

We want to prove the following.

Theorem 4.1 *Let C and T be a cube and a regular tetrahedron of the same volume. Suppose that each of them is cut into the same number of pieces by planes. (That is, we cut our polyhedron into two pieces, then cut one of the two pieces into two pieces, then cut one of the three pieces into two pieces, and so on.) It is not possible that the two collection of (polyhedral) pieces are the same.*

Proof. Let $\ell_1, \ldots, \ell_N$ be the lengths of all edges of all polyhedra involved in the two cutting processes. Let $\varphi_1, \ldots, \varphi_N$ are corresponding dihedral angles (we suppose that $0 < \varphi_i < \pi$ for all i). Take the sequence $\ell_1, \ldots, ell_N$ and remove from it any term which is a rational linear combination of the previous terms; we obtain a sequence $a_1, \ldots, a_m$ such that each of the ℓ_k's is equal to a unique rational linear combination of a_i's. Then do the same with the sequence $\pi, \varphi_1, \ldots, \varphi_N$; the resulting sequence is denoted as $\alpha_0 = \pi, \alpha_1, \ldots, \alpha_n$, and each of φ_k's is equal to a unique linear combination of α_j's. Call a convex polyhedron admissible, if the length of every edge is a rational linear combination of $a_1, \ldots, a_m$ and each dihedral angle is a rational linear combination of $\alpha_0, \alpha_1, \ldots, \alpha_n$.

Let $m_1, \ldots, m_q$ be the lengths of edges of an admissible convex polyhedron P, and let $\psi_1, \ldots, \psi_q$ be the corresponding dihedral angles. Let

$m_k = \sum_{i=1}^{m} r_{ki} a_i$ and $\psi_k = \sum_{j=0}^{n} s_{kj} \alpha_j$. Define the symbol of P by the formula

$$\mathrm{Symb}(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\sum_{k=1}^{q} r_{ki} s_{kj} \right) a_i \otimes \alpha_j.$$

Important remark: it is not a misprint that the second summation is taken from $j = 1$ to n, not from $j = 0$ to n; we do not include into the symbol the summand $s_{k0}\pi$. Thus, if one changes an angle by a rational multiple of π, then the symbol is not affected; if some dihedral angle *is* a rational multiple of π, then the corresponding edge do not appear in the expression for the symbol at all.

Example: the symbol of a cube (or of a rectangular box) is zero. Indeed, all the angles are $\pi/2$.

Exercise: the symbol of any rectangular prism with a polygonal base is zero.

Lemma 1 *Let P be a convex polyhedron. Suppose that it is cut by a plane L into two pieces, P' and P''. Then (provided that P, P', and P'' are admissible,*

$$\mathrm{Symb}(P) = \mathrm{Symb}(P') + \mathrm{Symb}(P'').$$

Proof of Lemma. Let $S = \{e_1, \ldots, e_q\}$ be the set of all edges of P, let ℓ_k be the length of the edge e_k and ψ_k be the corresponding dihedral angle. We divide the set S into four subsets: S_1 consists of edges which have no interior points in L and lie on the P' side of L; S_2 is the similar set with P'' instead of P'; S_3 consists of edges e_k cut by L into an edge e'_k of P' and an edge e''_k of P''; and S_4 consists of edges which are totally contained in L; for each $e_k \in S_4$, the dihedral angle ψ_k is divided by L into two parts: ψ'_k and ψ''_k. Consider also the intersection $L \cap P$. This is a convex polygon; each $e_k \in S_4$ is its side; let $T = \{f_1, \ldots, f_p\}$ be the set of all other sides of R. Each f_k is a side both of P' and P''; let m_k be the length of f_k and χ'_k, χ''_k be the corresponding dihedral angles in P' and P''. Obviously, $\chi'_k + \chi''_k = \pi$.

Edges of P':

– the edges $e_k \in S_1$; the lengths are ℓ_k, the angles are ψ_k;
– the edges e'_k for $e_k \in S_3$; the lengths are ℓ'_k, the angles are ψ_k;
– the edges $e_k \in S_4$; the lengths are ℓ_k, the angles are ψ'_k;
– the edges $f_k \in T$; the lengths are m_k, the angles are χ'_k.

Edges of P'':
– the edges $e_k \in S_2$; the lengths are ℓ_k, the angles are ψ_k;
– the edges e''_k for $e_k \in S_3$; the lengths are ℓ''_k, the angles are ψ_k;
– the edges $e_k \in S_4$; the lengths are ℓ_k, the angles are ψ''_k;
– the edges $f_k \in T$; the lengths are m_k, the angles are χ''_k.

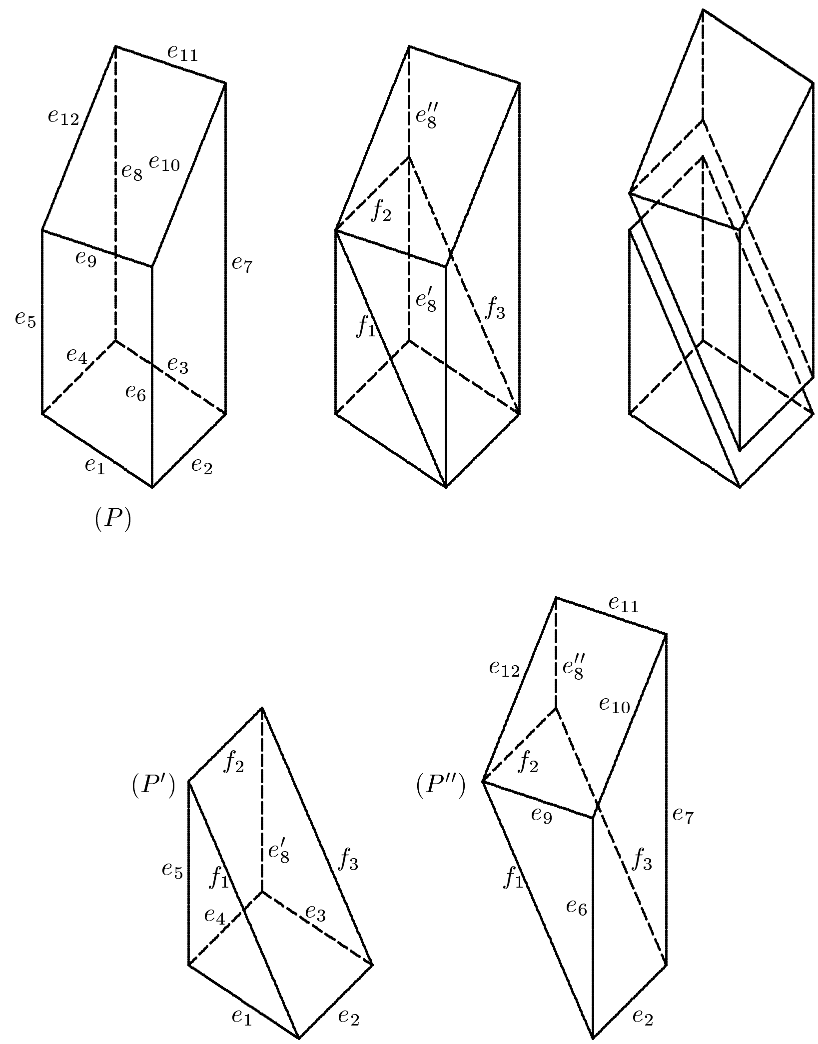

Figure 4:

The symbols of each of the polyhedra P', P'', and P consists of four groups of summands; for P' and P'' these groups correspond to the four groups of edges as listed above; for P they correspond to the sets S_1, S_2, S_3, S_4. The first group of summands in Symb(P') is the same as the first group of summands in Symb(P). The first group of summands in Symb(P'') is the same as the second group of summands in Symb(P). The sum of the second groups of summands in Symb(P') and Symb(P'') is the third group of summands in Symb(P) because $\ell'_k + \ell''_k = \ell_k$. The sum of the third groups of summands in Symb(P') and Symb(P'') is the fourth group of summands in Symb(P) because $\psi'_k + \psi''_k = \psi_k$. At last, the sum of the fourth groups of summands in Symb(P') and Symb(P'') is zero, since $\chi'_k + \chi''_k = \pi/2$. Thus, $\text{Symb}(P) = \text{Symb}(P') + \text{Symb}(P'')$ as stated by Lemma.

An example is shown on Figure 4. A polyhedron P (a four-gonal prism with non-parallel bases, shown at the left of the first row) is cut into two polyhedra by a plane (the cut is shown in the first row, the polyhedra P' and P'' are shown in the second row). The edges of P are $e_1, \dots e_{12}$; the sets S_i are: $S_1 = \{e_1, e_3, e_4, e_5\}, S_2 = \{e_6, e_7, e_9, e_{10}, e_{11}, e_{12}\}, S_3 = \{e_8\}, S_4 = \{e_2\}$.

Bach to Theorem. If two polyhedra can be cut into the same collection of polyhedral parts, then their symbols are both equal to the sum of the symbols of the part, and, hence, the symbols of the given two polyhedra are equal to each other. But the symbol of a cube is equal to zero, since all the angles are $\pi/2$ (see Example above). The symbol of a regular tetrahedron is equal to $6(\ell \otimes \alpha)$ where ℓ is the length of the edge and α is the dihedral angle. All we need to check is that α is not a rational multiple of π.

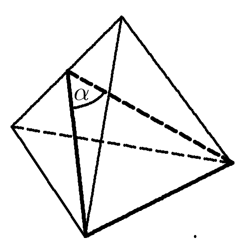

Figure 5:

The dihedral angle of a regular tetrahedron is the largest angle of an isosceles triangle whose sides are $\ell, \ell\frac{\sqrt{3}}{2}, \ell\frac{\sqrt{3}}{2}$ (see Figure 5). The cosine theorem shows that

$$\cos\alpha = \frac{\left(\ell\frac{\sqrt{3}}{2}\right)^2 \left(\ell\frac{\sqrt{3}}{2}\right)^2 + -\ell^2}{2\left(\ell\frac{\sqrt{3}}{2}\right)\left(\ell\frac{\sqrt{3}}{2}\right)} = \frac{1}{3}$$

Lemma 2 *If* $\cos\alpha = \frac{1}{3}$, *then* $\frac{\alpha}{\pi}$ *is irrational.*

Proof of Lemma. Otherwise, $\cos n\alpha = 1$ for some n. However, it is known from trigonometry that

$$\cos n\alpha = P_n(\cos\alpha)$$

where P_n is a polynomial of degree n with the leading coefficient 2^{n-1}.

[Proof by induction. Statement: for all n,

$$\cos n\alpha = P_n(\cos\alpha),\ \sin n\alpha = Q_n(\cos\alpha)\cdot\sin\alpha$$

where $\deg P_n = n$, $\deg Q_n = n-1$, and the leading coefficients of both P_n and Q_n are equal to 2^{n-1}. For $n=1$, this is true $(P_1(t) = t, Q_1(t) = 1)$; assume that the statement is true for some n. Then

$$\begin{aligned}\cos(n+1)\alpha &= \cos n\alpha\sin\alpha - \sin n\alpha\sin\alpha\\ &= P_n(cos\alpha)\cos\alpha - Q_n(cos\alpha)\sin^2\alpha\\ &= P_n(cos\alpha)\cos\alpha + Q_n(cos\alpha)(\cos^2\alpha - 1);\end{aligned}$$

$$\begin{aligned}\sin(n+1)\alpha &= \sin n\alpha\cos\alpha + \cos n\alpha\sin\alpha\\ &= Q_n(\cos\alpha)\sin\alpha\cos\alpha + P_n(\cos\alpha)\sin\alpha\\ &= (Q_n(\cos\alpha)\cos\alpha + P_n(\cos\alpha))\sin\alpha\end{aligned}$$

Hence,

$$\begin{aligned}P_{n+1}(t) &= P_n(t)t + Q_n(t)(t^2-1),\\ Q_{n+1}(t) &= Q_n(t)t + P_n(t),\end{aligned}$$

and the statement for the degrees and leading terms follows.]

This shows that

$$\cos n\alpha = P_n\left(\frac{1}{3}\right) = \frac{2^{n-1}}{3^n} + \frac{\text{an integer}}{3^{n-1}}$$

which cannot be an integer, in particular, 1.

This proves Lemma and completes the proof of Dehn's theorem.

5 Some further results.

In the language of algebra (which may technically not familiar to the reader, but the formulas below seem to me self-explanatory), the construction of the previous section assigns to every convex (actually, not necessarily convex) polyhedron a certain invariant, "Dehn's symbol",

$$\mathrm{Symb}(P) \in \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\pi\mathbb{Q}),$$

and Dehn's theorem states that if two polyhedra, P_1 and P_2, are equipartite (that is, can be cut by planes into identical collections of parts), then

$$\mathrm{Symb}(P_1) = \mathrm{Symb}(P_2).$$

(This is precisely the result of the previous section.)

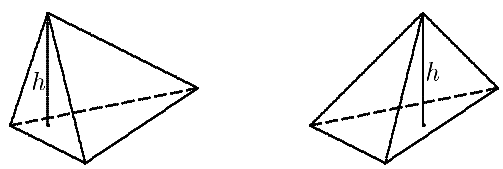

Figure 6:

Certainly, this may be applied not only to cubes and tetrahedra. The initial Hilbert's problem, by the way, dealt with a different example; Hilbert conjectured that two tetrahedra with equal bases and equal heights (like those on Figure 6) are not equipartite.

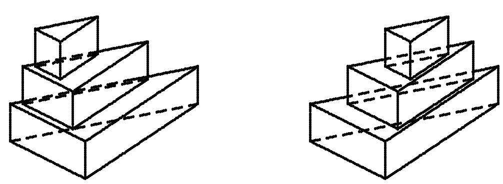

Figure 7:

The origin of this question belongs to the foundations of geometry. The whole theory of volumes of solids is based on the lemma stating that the volumes of tetrahedra in Figure 6 are the same. The similar planar lemma (involving the areas of triangles) has a direct geometric proof based on cutting and pasting. But the three-dimensional fact requires a limit "stair construction" involving pictures like Figure 7 (you can find a figure like this in textbooks in the spatial geometry). The question is, is this really necessary, and the answer is "yes": Dehn's theorem easily implies that the tetrahedra like those in Figure 7 are not, in general, equipartite.

More than 60 years after Dehn's work, Sydler proved that polyhedra with equal volumes and equal Dehn's invariant are equipartite [4]. There are similar results in spherical and hyperbolic geometries.

Dehn's symbol may be generalized to polyhedra of any dimension: for an n-dimensional polyhedron P,

$$\mathrm{Symb}(P) = \sum_{\substack{(n-2)\text{-dimensional}\\ \text{faces } s \text{ of } P}} \mathrm{volume(s)} \otimes \begin{bmatrix} \text{dihedral} \\ \text{angle at } s \end{bmatrix} \in \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\pi\mathbb{Q})$$

(the angle is formed by the two $(n-1)$-dimensional faces of P attached to s). In dimension 4, like in dimension 3, two polyhedra are equipartite, if and only if their volumes and their symbols are the same. But in dimension 5 it is not true any longer: there arises a new invariant, a "secondary Dehn's symbol" involving a summation over the edges (for an n-dimensional polyhedron, over $(n-4)$-dimensional faces) of P. There is a conjecture (I do not know its current status) that an "equipartite type" of an n-dimensional polyhedron is characterized by a sequence of $\left[\frac{n+1}{2}\right]$ invariants: the volume, Dehn's symbol, secondary Dehn's symbol, and so on, taking values in more and more complicated tensor products (k-th Dehn symbol involves a summation over $(n-2k)$-dimensional faces. In particular, for one- and two-dimensional polyhedra (segments and polygons) only the "volume" (the length and the area) counts; in dimensions 3 and 4 we also have Dehn's symbol), and so on.

If you want to know more about this, you can read, in addition to the popular book of Boltianskii, the article of Cartier in Proceedings of the Bourbaki Seminar [2]. But I am not sure that it has been ever translated from French into English, so if you read this article, you have a chance to study a beautiful language, in addition to a beautiful geometry.

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