Berkeley Math Circle

INTRODUCTION TO GAME THEORY

by Zvezdelina Stankova
Mills College and
University of California at Berleley
September 12, 2004

Contents

Note: The problems in this handout are compiled from the book *Mathematical Circles, the Russian experience*, by Fomin, Genkin and Itenberg, published by the American Mathematical Society, 1996, www.ams.org.

In the majority of the following games, two players take turns making moves. Our goal is to find out if there is a strategy for Player I or Player II to *always win*, and if the answer is yes, to devise such a strategy.

1. Chess Games

(1) On a chessboard, a rook stands on square $a1$ (the bottom left corner). Players take turns moving the rook as many squares as they want, either horizontally to the right or vertically upward. The player who can place the rook on square $h8$ (top right corner) wins. [1]

(2) A king is placed on square $a1$ of a chessboard. Players take turns moving the king either upwards, to the right, or along a diagonal going northeast. The player who places the king on square $h8$ is the winner. [2]

(3) A queen stands on square $c1$ (third cell from left to right in the bottom row) of a chessboard. Players take turns moving the queen any number of squares to the right, upwards, or along a diagonal in the northeast direction. The player who can place the queen in square $h8$ wins.[3]

(4) A knight is placed on square $a1$ of a chessboard. Players alternate moving the knight either two squares to the right and one square up or down, or two squares up and one square right or left. The player who cannot move loses.[4]

[1] In chess, a *rook* can move either horizontally or vertically, as many squares as the player wants. The rook in our game has a restriction: it can move only horizontally to the right or vertically up, so that there is no back-tracking, and eventually, the rook ends up in the top right corner and the game ends.

[2] In chess, a *king* can move to any of the adjacent 8 squares: 4 to the sides and 4 diagonally. The king in our problem is "disabled": he can move only in north, east or northeast directions, so that eventually he ends up in the top right corner and the game ends.

[3] In chess, a *queen* can move horizontally, vertically or diagonally, as many squares as the player wants. The queen in our game has a restriction: she can move only horizontally to the east, vertically to the north, or diagonally to the northeast, so that there is no back-tracking, and eventually, she ends up in the top right corner and the game ends.

[4] In chess, a *knight* moves 2 squares in one direction (horizontally/vertically) and 1 square in the other direction (vertically/horizontally). Tracing out all possibilities, this gives at most 8 moves. Our knight can move only in 4 of those positions.

2. Games or Graph Theory?

(5) 4 knights are situated on a 3×3 chessboard as follows: a white knight is on each square $c1$ and $c3$, and a black knight is on each squares $a1$ and $a3$. Can the knights move, using the usual chess knight's move, to the following final position: a white knight on square $c1$, a black knight on $c3$, a black knight on $a1$ and a white knight on $a3$?

(6) 3 pawns are placed on the vertices of a pentagon. It is allowed to move a pawn along any diagonal of the pentagon to any free vertex. Is it possible that after several such moves one of the pawns occupies its original position while the other two have changed their places?

3. Games with Checkers, Piles of Candy, Stones, Matches and What-Not

(7) A checker is placed at each end of a strip of squares measuring 1×20. Players take turns moving either checker in the direction of the other, each by one or by two squares. A checker cannot jump over another checker. The player who cannot move loses.

(8) There are two piles of candy. One contains 20 pieces, and the other 21. Players take turns eating all the candy in one pile, and separating the remaining candy into two (not necessarily equal) non-empty piles. The player who cannot move loses.

(9) Of two piles of stones, one contains 7 stones, and the other 5. Players alternate taking any number of stones from one of the piles, or an equal number from each pile. The player who cannot move loses.

(10) (a) There are two piles of 7 stones each. In each turn, a player may take a single stone from one of the piles, or a stone from each pile. The player who cannot move loses.

(b) In addition to the moved described in part (a), players are allowed to take a stone from the first pile and place it on the second pile. Other rules remain the same.

(11) There are three piles of stones. The first contains 50 stones, the second 60 stones, and the third 70. A turn consists in dividing each of the piles containing more than one stone into two smaller piles. The player who leaves piles of individual stones is the winner.

(12) A box contains 300 matches. Players take turns removing no more than half of the matches in the box. The player who cannot move loses.

(13) There are two piles of matches:

(a) a pile of 101 matches and a pile of 201 matches;

(b) a pile of 100 matches and a pile of 201 matches.

Players take turns removing a number of matches from one pile which is equal to one of the divisors of the number of matches in the other pile. The player removing the last match wins.[5]

(14) There are two piles of 11 matches each. In one turn, a player must take two matches from one pile and one match from the other. The player who cannot move loses.

---

[5] A number a is called a *divisor* of a number b if we can divide b by a without a remainder. For example, 4 is a divisor of 8 since $8/4 = 2$, but 4 is *not* a divisor of 11 since $11/4$ is 2 with remainder 3, or in other words, $11/4 = 2.75$ is not a whole number.

4. Games with Numbers

(15) The number 60 is written on a blackboard. Players take turns subtracting from the number on the blackboard any of its divisors, and replacing the original number with the result of this subtraction. The player who writes the number 0 loses.

(16) This game begins with the number 0. In one turn a player can add to the current number any natural number from 1 through 9. The player who reaches the number 100 wins.

(17) This game begins with the number 1. In one turn a player can add to the current number any natural number from 2 through 9. The player who reaches a number greater than 1000 wins.

(18) The game begins with a the number 2. In one turn, a player can add to the current number any natural number smaller than it. The player who reaches the number 1000 wins.

(19) This game begins with a the number 1000. In one turn, a player can subtract from the current number any natural number less than it which is a power of 2 (e.g. $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, $32 = 2^5$, $64 = 2^6$, etc.) The player who reaches the number 0 wins.

5. How to Become A Millionaire?

(20) On the TV show "Fortune of Mathematics", a contest is held among several players. Each player initially has a heap of 100 stones; the player divides the heap into two parts, then divides one of the parts into two again, et cetera, until the player has 100 separate stones. After each division, the player records the product of the numbers of stones in the two new heaps, and at the end the player adds up all of these products. The player with the largest final sum N wins the game and receives an award of $\$N \times 201$. The players can see each others moves **after** a move has been made, and the TV host makes sure that everyone completes his/her next move BEFORE letting everyone see each other's positions. Is there a strategy for one of the players to become a millionaire? Note that draws of highest sums do **NOT** win anyone any money.