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PENTAGON, ICOSAHEDRON, ... WHAT'S NEXT?

Part I

Homework problems.

Problem 1. Let A be the area of a regular pentagon. Continue the sides of the pentagon to form a larger, star-shaped region and compute the added area.

Problem 2. Consider the space $\mathbf{R}^n$ of all real-valued functions $f : \{1, 2, ..., n\} \to \mathbf{R}$ on the finite set $\{1, 2, ..., n\}$. Define the distance between two such functions, f and g, by the formula:

$$d(f, g) = \max_{k=1,2,...,n} |f(k) - g(k)|.$$

Show that the distance satisfies the triangle inequality:

$$d(f, h) \leq d(f, g) + d(g, h) \text{ for any } f, g, h.$$

Define the "unit ball" with respect to this distance and describe it explicitly.

Do the same — for the distance defined by another formula:

$$d(f, g) = |f(1) - g(1)| + ... + |f(n) - g(n)|.$$

Can you define the notion of distance between functions when the finite domain set is replaced by the continuous interval $I = [0, 1]$ of real numbers?

Problem 3. Convince yourself [1] that any transformation of the Euclidean $3D$-space which preserves all distances between points (with the usual concept of distance!) and preserves the origin is either a rotation about an axis passing through the origin or the composition of such a rotation with the central symmetry about the origin. Check that central symmetry commutes with all the other transformations. Show that the composition of reflections in two diffrent mirrors is a rotation. How about the composition of three different mirror reflections?

[1] I don't know a rigorous elementary proof of this statement.

Problem 4. Let σ be a permutation on the set $\{1, 2, ..., n\}$ (*i.e.* an invertible function from this set to itself). Define the *length* $l(\sigma)$ of the permutation as the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$.

(a) What is the maximal possible value of the length?

(b) Compose σ with a *transposition* τ_{ab} (*i.e.* the swap of two indices a and b) of two nearby indices $a = i, b = i + 1$ and show that

$$l(\sigma \cdot \tau) = l(\sigma) \pm 1.$$

What does the sign $\pm$ in this formula depend on?

(c) Prove that any permutation of odd (even) length can be obtained as the composition of transpositions whose number will be necessarily odd (respectively even).

(d) Show that the minimal number of transpositions needed for representing σ equals $l(\sigma)$.

(e) Define the *sign* of a pemutation by $sign(\sigma) = (-1)^{l(\sigma)}$ and prove that for any permutations

$$sign(\sigma_1 \cdot \sigma_2) = sign(\sigma_1)\ sign(\sigma_2).$$

Problem 5. Let S_n denote the group [2] of all permutations on $\{1, ..., n\}$, the operation being composition of permutations.

(a) Construct a homomorphism [3] of S_4 onto S_3.

Hint: think of S_4 as the symmetry group of the tetrahedron.

(b) Identify the *rotation* group of the cube with S_4 (and give another solution to part (a)).

Hint: consider the action of the *symmetry* group of the cube on the four diagonals and find out which of the symmetries preserve all the diagonals.

(c) Identify the *rotation* group of the dodecahedron with the group of all *even* permutations on $\{1, 2, 3, 4, 5\}$.

[2] By definition, a *group* is a set G equipped with an operation $a, b \mapsto ab$ satisfying
(i) $ab \in G$ for any $a, b \in G$,
(ii) $(ab)c = a(bc)$,
(iii) there exists a unique $e \in G$ such that $ae = ea = a$ for all $a \in G$,
(iv) for any $a \in G$ there exists a unique $a^{-1} \in G$ such that $aa^{-1} = e = a^{-1}a$.

[3] A group *homomorphism* between two groups is a function $G \to G'$ which respects the operations, *i.e.* if $a \mapsto a'$ and $b \mapsto b'$ then $ab \mapsto a'b'$.

Hint: examine the way the *symmetries* of the dodecahedron act on the five cubes formed by the diagonals in the dodecahedron's faces.

Problem 6. Recall that the *n-dimensional simplex* is defined as the set of all points in the $n+1$-dimensional space with coordinates $x_0, x_1, ..., x_n$ satisfying

$$x_0 + x_1 + ... + x_n = 1, \quad x_0 \geq 0, x_1 \geq 0, ..., x_n \geq 0.$$

The *n-dimensional cube* is defined as the set of points in the n-dimensional space with coordinates $x_1, ..., x_n$ satisfying the inequaltities $-1 \leq x_i \leq 1$ for each $i = 1, ..., n$.

(a) Show that for $n = 1, 2, 3$ the n-dimensional simplex is respectively the segment, the regular triangle, the tetrahedron.

(b) Show that all k-dimensional faces (*i.e.* vertices, edges, etc.) of the n-dimensional simplex are simplexes of dimension $k = 0, 1, ..., n-1$ and find the number of such faces for each k.

Problem 7. The *n-dimensional cube* is defined as the set of points in the n-dimensional space with coordinates $x_1, ..., x_n$ satisfying the inequaltities $-1 \leq x_i \leq 1$ for each $i = 1, ..., n$.

(a) Formulate and solve the version Problem 5 with the simplexes replaces by the cubes.

(b) Show that pemutations of coordinates, changes of signs of one or several coordinates or compositions of these transformations preserve the n-dimensional cube. Find the number of elements in the symmetry group of the n-dimensional cube formed by these transformations.

(c) Paint black and white the vertices $(\pm 1, ..., \pm 1)$ of the n-dimensional cube which have respectively even and odd number of -1's. How many transformations from the symmetry group of the cube keep black and white vertices apart?

(d) Show that for n odd the black and white vertices form two polyhedra centrally symmetric to each other. What are the polyherda in the case $n = 3$?

(e) Show that for n even the black (respectively white) vertices form a cenntrally symmetric polyhedron. Study the case $n = 4$ and show that the black polyhedron can be identified with the 4-dimensional version of the octahedron (defined as the polyhedron formed by centers of a cube's faces). Is the same true for the polyhedron formed by the white vertices?