LINEAR RECURSIVE SEQUENCES

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1. Sequences

A sequence is an infinite list of numbers, like

\( 1, 2, 4, 8, 16, 32, \ldots \)\n
The numbers in the sequence are called its terms. The general form of a sequence is

\( a_1, a_2, a_3, \ldots \)

where \( a_n \) is the \( n \)-th term of the sequence. In the example (1) above, \( a_1 = 1, a_2 = 2, a_3 = 4, \) and so on.

The notations \( \{a_n\} \) or \( \{a_n\}_{n=1}^{\infty} \) are abbreviations for

\( a_1, a_2, a_3, \ldots \)

Occasionally the indexing of the terms will start with something other than 1. For example, \( \{a_n\}_{n=0}^{\infty} \) would mean

\( a_0, a_1, a_2, \ldots \)

(In this case \( a_n \) would be the \((n + 1)\)-st term.)

For some sequences, it is possible to give an explicit formula for \( a_n \); this means that \( a_n \) is expressed as a function of \( n \). For instance, the sequence (1) above can be described by the explicit formula \( a_n = 2^{n-1} \).

2. Recursive definitions

An alternative way to describe a sequence is to list a few terms and to give a rule for computing the rest of the sequence. Our example (1) above can be described by the starting value \( a_1 = 1 \) and the rule \( a_{n+1} = 2a_n \) for integers \( n \geq 1 \). Starting from \( a_1 = 1 \), the rule implies that

\[
\begin{align*}
a_2 &= 2a_1 = 2(1) = 2 \\
a_3 &= 2a_2 = 2(2) = 4 \\
a_4 &= 2a_3 = 2(4) = 8,
\end{align*}
\]

and so on; each term in the sequence can be computed recursively in terms of the terms previously computed. A rule such as this giving the next term in terms of earlier terms is also called a recurrence relation (or simply recurrence).

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3. Linear recursive sequences

A sequence \( \{a_n\} \) is said to satisfy the \textit{linear recurrence} with coefficients \( c_k, c_{k-1}, \ldots, c_0 \) if
\[
c_k a_{n+k} + c_{k-1} a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0
\]
holds for all integers \( n \) for which this makes sense. (If the sequence starts with \( a_1 \), then this means for \( n \geq 1 \).) The integer \( k \) is called the \textit{order} of the linear recurrence.

A \textit{linear recursive sequence} is a sequence of numbers \( a_1, a_2, a_3, \ldots \) satisfying some linear recurrence as above with \( c_k \neq 0 \) and \( c_0 \neq 0 \). For example, the sequence (1) satisfies
\[
a_{n+1} - 2a_n = 0
\]
for all integers \( n \geq 1 \), so it is a linear recursive sequence satisfying a recurrence of order 1, with \( c_1 = 1 \) and \( c_0 = -2 \).

Requiring \( c_k \neq 0 \) guarantees that the linear recurrence can be used to express \( a_{n+k} \) as a linear combination of earlier terms:
\[
a_{n+k} = -\frac{c_{k-1}}{c_k} a_{n+k-1} - \cdots - \frac{c_1}{c_k} a_{n+1} - \frac{c_0}{c_k} a_n.
\]
The requirement \( c_0 \neq 0 \) lets one express \( a_n \) as a linear combination of later terms:
\[
a_n = -\frac{c_k}{c_0} a_{n+k} - \frac{c_{k-1}}{c_0} a_{n+k-1} - \cdots - \frac{c_1}{c_0} a_{n+1}.
\]
This lets one define \( a_0, a_{-1}, \) and so on, to obtain a \textit{doubly infinite sequence}
\[
\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots
\]
that now satisfies the same linear recurrence for all integers \( n \), positive or negative.

4. Characteristic polynomials

The \textit{characteristic polynomial} of a linear recurrence
\[
c_k a_{n+k} + c_{k-1} a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0
\]
is defined to be the polynomial
\[
c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0.
\]
For example, the characteristic polynomial of the recurrence \( a_{n+1} - 2a_n = 0 \) satisfied by the sequence (1) is \( x - 2 \).

Here is another example: the famous \textit{Fibonacci sequence}
\[
\{F_n\}_{n=0}^{\infty} = 0, 1, 1, 2, 3, 5, 8, 13, \ldots
\]
which can be described by the starting values \( F_0 = 0, F_1 = 1 \) and the recurrence relation
\[
F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 2.
\]
To find the characteristic polynomial, we first need to rewrite the recurrence relation in the form (2). The relation (3) is equivalent to
\[
F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.
\]
Rewriting it as
\[
F_{n+2} - F_{n+1} - F_n = 0
\]
shows that \( \{F_n\} \) is a linear recursive sequence satisfying a recurrence of order 2, with \( c_2 = 1, c_1 = -1, \) and \( c_0 = -1 \). The characteristic polynomial is \( x^2 - x - 1 \).
5. IDEALS AND MINIMAL CHARACTERISTIC POLYNOMIALS

The same sequence can satisfy many different linear recurrences. For example, doubling (5) shows the Fibonacci sequence also satisfies
\[ 2F_{n+2} - 2F_{n+1} - 2F_n = 0, \]
which is a linear recurrence with characteristic polynomial \( 2x^2 - 2x - 2 \). It also satisfies
\[ F_{n+3} - F_{n+2} - F_{n+1} = 0, \]
and adding these two relations, we find that \( \{F_n\} \) also satisfies
\[ F_{n+3} + F_{n+2} - 3F_{n+1} - 2F_n = 0 \]
which has characteristic polynomial \( x^3 + x^2 - 3x - 2 = (x + 2)(x^2 - x - 1) \).

Now consider an arbitrary sequence \( \{a_n\} \). Let \( I \) be the set of characteristic polynomials of all linear recurrences satisfied by \( \{a_n\} \). Then

(a) If \( f(x) \in I \) and \( g(x) \in I \) then \( f(x) + g(x) \in I \).
(b) If \( f(x) \in I \) and \( h(x) \) is any polynomial, then \( h(x)f(x) \in I \).

In general, a nonempty set \( I \) of polynomials satisfying (a) and (b) is called an ideal.

**Fact from algebra:** Let \( I \) be an ideal of polynomials. Then either \( I = \{0\} \) or else there is a unique monic polynomial \( f(x) \in I \) such that
\[ I = \text{the set of polynomial multiples of } f(x) = \{ h(x)f(x) \mid h(x) \text{ is a polynomial} \}. \]
(A polynomial is monic if the coefficient of the highest power of \( x \) is 1.)

This fact, applied to the ideal of characteristic polynomials of a linear recursive sequence \( \{a_n\} \) shows that there is always a minimal characteristic polynomial \( f(x) \), which is the monic polynomial of lowest degree in \( I \). It is the characteristic polynomial of the lowest order non-trivial linear recurrence satisfied by \( \{a_n\} \). The characteristic polynomial of any other linear recurrence satisfied by \( \{a_n\} \) is a polynomial multiple of \( f(x) \).

The order of a linear recursive sequence \( \{a_n\} \) is defined to be the lowest order among all (nontrivial) linear recurrences satisfied by \( \{a_n\} \). The order also equals the degree of the minimal characteristic polynomial. For example, as we showed above, \( \{F_n\} \) satisfies
\[ F_{n+3} + F_{n+2} - 3F_{n+1} - 2F_n = 0, \]
but we also know that
\[ F_{n+2} - F_{n+1} - F_n = 0, \]
and it is easy to show that \( \{F_n\} \) cannot satisfy a linear recurrence of order less than 2, so \( \{F_n\} \) is a linear recursive sequence of order 2, with minimal characteristic polynomial \( x^2 - x - 1 \).

6. THE MAIN THEOREM

**Theorem 1.** Let \( f(x) = c_kx^k + \cdots + c_0 \) be a polynomial with \( c_k \neq 0 \) and \( c_0 \neq 0 \). Factor \( f(x) \) over the complex numbers as
\[ f(x) = c_k(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_\ell)^{m_\ell}, \]
where \( r_1, r_2, \ldots, r_\ell \) are distinct nonzero complex numbers, and \( m_1, m_2, \ldots, m_\ell \) are positive integers. Then a sequence \( \{a_n\} \) satisfies the linear recurrence with characteristic polynomial \( f(x) \) if and only if there exist polynomials \( g_1(n), g_2(n), \ldots, g_\ell(n) \) with \( \deg g_i \leq m_i - 1 \) for \( i = 1, 2, \ldots, \ell \) such that

\[
a_n = g_1(n)r_1^n + \cdots + g_\ell(n)r_\ell^n \quad \text{for all } n.
\]

Here is an important special case.

**Corollary 2.** Suppose in addition that \( f(x) \) has no repeated factors; in other words suppose that \( m_1 = m_2 = \cdots = m_\ell = 1 \). Then \( f(x) = c_k(x - r_1)(x - r_2)\cdots(x - r_\ell) \) where \( r_1, r_2, \ldots, r_\ell \) are distinct nonzero complex numbers (the roots of \( f \)). Then \( \{a_n\} \) satisfies the linear recurrence with characteristic polynomial \( f(x) \) if and only if there exist constants \( B_1, B_2, \ldots, B_\ell \) (not depending on \( n \)) such that

\[
a_n = B_1 r_1^n + \cdots + B_\ell r_\ell^n \quad \text{for all } n.
\]

7. **Example: solving a linear recurrence**

Suppose we want to find an explicit formula for the sequence \( \{a_n\} \) satisfying \( a_0 = 1 \), \( a_1 = 4 \), and

\[
a_{n+2} = \frac{a_{n+1} + a_n}{2} \quad \text{for } n \geq 0.
\]

Since \( \{a_n\} \) satisfies a linear recurrence with characteristic polynomial \( x^2 - \frac{1}{2}x - \frac{1}{2} = (x - 1)(x + \frac{1}{2}) \), we know that there exist constants \( A \) and \( B \) such that

\[
a_n = A(1)^n + B \left( -\frac{1}{2} \right)^n
\]

for all \( n \). The formula (7) is called the *general solution* to the linear recurrence (6). To find the *particular solution* with the correct values of \( A \) and \( B \), we use the known values of \( a_0 \) and \( a_1 \):

\[
1 = a_0 = A(1)^0 + B \left( -\frac{1}{2} \right)^0 = A + B
\]

\[
4 = a_1 = A(1)^1 + B \left( -\frac{1}{2} \right)^1 = A - B/2.
\]

Solving this system of equations yields \( A = 3 \) and \( B = -2 \). Thus the particular solution is

\[
a_n = 3 - 2 \left( -\frac{1}{2} \right)^n.
\]

(As a check, one can try plugging in \( n = 0 \) or \( n = 1 \).)

8. **Example: the formula for the Fibonacci sequence**

As we worked out earlier, \( \{F_n\} \) satisfies a linear recurrence with characteristic polynomial \( x^2 - x - 1 \). By the quadratic formula, this factors as \( (x - \alpha)(x - \beta) \) where \( \alpha = (1 + \sqrt{5})/2 \) is the golden ratio, and \( \beta = (1 - \sqrt{5})/2 \). The main theorem implies that there are constants \( A \) and \( B \) such that

\[
F_n = A\alpha^n + B\beta^n
\]
for all \( n \). Using \( F_0 = 0 \) and \( F_1 = 1 \) we obtain
\[
0 = A + B, \quad 1 = A\alpha + B\beta.
\]
Solving for \( A \) and \( B \) yields \( A = 1/(\alpha - \beta) \) and \( B = -1/(\alpha - \beta) \), so
\[
F_n = \alpha^n - \beta^n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]
\]
for all \( n \).

9. Example: finding a linear recurrence from an explicit formula

Let \( a_n = (n + 2^n)F_n \), where \( \{F_n\} \) is the Fibonacci sequence. Then by the explicit formula for \( F_n \),
\[
a_n = (n + 2^n) \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) = \left[ \left( \frac{1}{\alpha - \beta} \right) n \right] \alpha^n + \left[ \left( \frac{-1}{\alpha - \beta} \right) n \right] \beta^n + \left( \frac{1}{\alpha - \beta} \right) (2\alpha)^n + \left( \frac{-1}{\alpha - \beta} \right) (2\beta)^n.
\]
By Theorem 1, \( \{a_n\} \) satisfies a linear recurrence with characteristic polynomial
\[
(x - \alpha)^2(x - \beta)^2(x - 2\alpha)(x - 2\beta) = (x^2 - x - 1)^2 \left[ x^2 - 2(\alpha + \beta) + 4\alpha\beta \right]
= (x^2 - x - 1)^2(x^2 - 2x - 4)
= x^6 - 4x^5 - x^4 + 12x^3 + x^2 - 10x + 4,
\]
where we have used the identity \( x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - x - 1 \) to compute \( \alpha + \beta \) and \( \alpha\beta \). In other words,
\[
a_{n+6} - 4a_{n+5} - a_{n+4} + 12a_{n+3} + a_{n+2} - 10a_{n+1} + 4a_n = 0
\]
for all \( n \). In fact, we have found the minimal characteristic polynomial, since if the actual minimal characteristic polynomial were a proper divisor of \( (x^2 - x - 1)^2(x^2 - 2x - 4) \), then according to Theorem 1, the explicit formula for \( a_n \) would have had a different, simpler form.

10. Inhomogeneous recurrence relations

Suppose we wanted an explicit formula for a sequence \( \{a_n\} \) satisfying \( a_0 = 0 \), and
\[
a_{n+1} - 2a_n = F_n \quad \text{for} \quad n \geq 0,
\]
where \( \{F_n\} \) is the Fibonacci sequence as usual. This is not a linear recurrence in the sense we have been talking about (because of the \( F_n \) on the right hand side instead of 0), so our usual method does not work. A recurrence of this type, linear except for a function of \( n \) on the right hand side, is called an inhomogeneous recurrence.

We can solve inhomogeneous recurrences explicitly when the right hand side is itself a linear recursive sequence. In our example, \( \{a_n\} \) also satisfies
\[
a_{n+2} - 2a_{n+1} = F_{n+1}
\]
and
\[
a_{n+3} - 2a_{n+2} = F_{n+2}.
\]
Subtracting (8) and (9) from (10) yields
\[ a_{n+3} - 3a_{n+2} + a_{n+1} + 2a_n = F_{n+2} - F_{n+1} - F_n = 0. \]
Thus \( \{a_n\} \) is a linear recursive sequence after all! The characteristic polynomial of this new linear recurrence is \( x^3 - 3x^2 + x + 2 = (x - 2)(x^2 - x - 1) \), so by Theorem 1, there exist constants \( A, B, C \) such that
\[ a_n = A \cdot 2^n + B\alpha^n + C\beta^n \]
for all \( n \). Now we can use \( a_0 = 0 \), and the values \( a_1 = 0 \) and \( a_2 = 1 \) obtained from (8) to determine \( A, B, C \). After some work, one finds \( A = 1, B = -\alpha^2/\alpha - \beta \), and \( C = \beta^2/(\alpha - \beta) \), so \( a_n = 2^n - F_{n+2} \).

If \( \{x_n\} \) is any other sequence satisfying
\[ x_{n+1} - 2x_n = F_n \]
but not necessarily \( x_0 = 0 \), then subtracting (8) from (11) shows that the sequence \( \{y_n\} \) defined by \( y_n = x_n - a_n \) satisfies \( y_{n+1} - 2y_n = 0 \) for all \( n \), so \( y_n = D \cdot 2^n \) for some number \( D \). Hence the general solution of (11) has the form
\[ x_n = 2^n - F_{n+2} + D \cdot 2^n, \]
or more simply,
\[ x_n = E \cdot 2^n - F_{n+2}, \]
where \( E \) is some constant.

In general, this sort of argument proves the following.

**Theorem 3.** Let \( \{b_n\} \) be a linear recursive sequence satisfying a recurrence with characteristic polynomial \( f(x) \). Let \( g(x) = c_kx^k + c_{k-1}x^{k-1} + \cdots + c_1x + c_0 \) be a polynomial. Then every solution \( \{x_n\} \) to the inhomogeneous recurrence
\[ c_kx_{n+k} + c_{k-1}x_{n+k-1} + \cdots + c_1x_{n+1} + c_0x_n = b_n \]
also satisfies a linear recurrence with characteristic polynomial \( f(x)g(x) \). Moreover, if \( \{x_n\} = \{a_n\} \) is one particular solution to (12), then all solutions have the form \( x_n = a_n + y_n \), where \( \{y_n\} \) ranges over the solutions of the linear recurrence
\[ c_ky_{n+k} + c_{k-1}y_{n+k-1} + \cdots + c_1y_{n+1} + c_0y_n = 0. \]

**11. The Mahler-Lech theorem**

Here is a deep theorem about linear recursive sequences:

**Theorem 4** (Mahler-Lech theorem). Let \( \{a_n\} \) be a linear recursive sequence of complex numbers, and let \( c \) be a complex number. Then there exists a finite (possibly empty) list of arithmetic progressions \( T_1, T_2, \ldots T_m \) and a finite (possibly empty) set \( S \) of integers such that
\[ \{ n \mid a_n = c \} = S \cup T_1 \cup T_2 \cup \cdots \cup T_m. \]

Warning: don’t try to prove this at home! This is very hard to prove. The proof uses “\( p \)-adic numbers.”
12. Problems

There are a lot of problems here. Just do the ones that interest you.

(1) If the Fibonacci sequence is extended to a doubly infinite sequence satisfying the same linear recurrence, then what will $F_{-4}$ be? (Is it easier to do this using the recurrence, or using the explicit formula?)

(2) Find the smallest degree polynomial that could be the minimal characteristic polynomial of a sequence that begins

$$2, 5, 18, 67, 250, 933, \ldots$$

Assuming that the sequence is a linear recursive sequence with this characteristic polynomial, find an explicit formula for the $n$-th term.

(3) Suppose that $a_n = n^2 + 3n + 7$ for $n \geq 1$. Prove that $\{a_n\}$ is a linear recursive sequence, and find its minimal characteristic polynomial.

(4) Suppose $a_1 = a_2 = a_3 = 1$, $a_4 = 3$, and $a_{n+4} = 3a_{n+2} - 2a_n$ for $n \geq 1$. Prove that $a_n = 1$ if and only if $n$ is odd or $n = 2$. (This is an instance of the Mahler-Lech theorem: for this sequence, one would take $S = \{2\}$ and $T_1 = \{1, 3, 5, 7, \ldots \}$.)

(5) Suppose $a_0 = 2$, $a_1 = 5$, and $a_{n+2} = (a_{n+1})^2(a_n)^3$ for $n \geq 0$. (This is a recurrence relation, but not a linear recurrence relation.) Find an explicit formula for $a_n$.

(6) Suppose $\{a_n\}$ is a sequence such that $a_{n+2} = a_{n+1} - a_n$ for all $n \geq 1$. Given that $a_{38} = 7$ and $a_{55} = 3$, find $a_1$. (Hint: it is possible to solve this problem with very little calculation.)

(7) Let $\theta$ be a fixed real number, and let $a_n = \cos(n\theta)$ for integers $n \geq 1$. Prove that $\{a_n\}$ is a linear recursive sequence, and find the minimal characteristic polynomial. (Hint: if you know the definition of $\cos x$ in terms of complex exponentials, use that. Otherwise, use the sum-to-product rule for the sum of cosines $\cos(n\theta) + \cos((n+2)\theta)$. For most but not all $\theta$, the degree of the minimal characteristic polynomial will be 2.)

(8) Give an example of a sequence that is not a linear recursive sequence, and prove that it is not one.

(9) Given a finite set $S$ of positive integers, show that there exists a linear recursive sequence

$$a_1, a_2, a_3, \ldots$$
such that $\{n \mid a_n = 0\} = S$.

(10) A student tosses a fair coin and scores one point for each head that turns up, and two points for each tail. Prove that the probability of the student scoring $n$ points at some time in a sequence of $n$ tosses is $\frac{1}{3} \left(2 + \left(-\frac{1}{3}\right)^n\right)$.

(11) Let $F_n$ denote the $n$-th Fibonacci number. Let $a_n = (F_n)^2$. Prove that $a_1, a_2, a_3, \ldots$ is a linear recursive sequence, and find its minimal characteristic polynomial.

(12) Prove the “fact from algebra” mentioned above in Section 5. (Hint: if $I \neq \{0\}$, pick a nonzero polynomial in $I$ of smallest degree, and multiply it by a constant to get a monic polynomial $f(x)$. Use long division of polynomials to show that anything else in $I$ is a polynomial multiple of $f(x)$.)

(13) Suppose that $a_1, a_2, \ldots$ is a linear recursive sequence. For $n \geq 1$, let $s_n = a_1 + a_2 + \cdots + a_n$. Prove that $\{s_n\}$ is a linear recursive sequence.

(14) Suppose $\{a_n\}$ and $\{b_n\}$ are linear recursive sequences. Let $c_n = a_n + b_n$ and $d_n = a_n b_n$ for $n \geq 1$.

(a) Prove that $\{c_n\}$ and $\{d_n\}$ also are linear recursive sequences.
(b) Suppose that the minimal characteristic polynomials for \( \{a_n\} \) and \( \{b_n\} \) are \( x^2 - x - 2 \) and \( x^2 - 5x + 6 \), respectively. What are the possibilities for the minimal characteristic polynomials of \( \{c_n\} \) and \( \{d_n\} \)?

(15) Suppose that \( \{a_n\} \) and \( \{b_n\} \) are linear recursive sequences. Prove that
\[
a_1, b_1, a_2, b_2, a_3, b_3, \ldots
\]
also is a linear recursive sequence.

(16) Use the Mahler-Lech theorem to prove the following generalization. Let \( \{a_n\} \) be a linear recursive sequence of complex numbers, and let \( p(x) \) be a polynomial. Then there exists a finite (possibly empty) list of arithmetic progressions \( T_1, T_2, \ldots, T_m \) and a finite (possibly empty) set \( S \) of integers such that
\[
\{ n \mid a_n = p(n) \} = S \cup T_1 \cup T_2 \cup \cdots \cup T_m.
\]
(Hint: let \( b_n = a_n - p(n) \).)

(17) (1973 USAMO, no. 2) Let \( \{X_n\} \) and \( \{Y_n\} \) denote two sequences of integers defined as follows:
\[
X_0 = 1, X_1 = 1, X_{n+1} = X_n + 2X_{n-1} \quad (n = 1, 2, 3, \ldots),
\]
\[
Y_0 = 1, Y_1 = 7, Y_{n+1} = 2Y_n + 3Y_{n-1} \quad (n = 1, 2, 3, \ldots).
\]
Thus, the first few terms of the sequences are:
\[
X : 1, 1, 3, 5, 11, 21, \ldots,
\]
\[
Y : 1, 7, 17, 55, 161, 487, \ldots.
\]
Prove that, except for the “1,” there is no term which occurs in both sequences.

(18) (1963 IMO, no. 4) Find all solutions \( x_1, x_2, x_3, x_4, x_5 \) to the system
\[
x_5 + x_2 = yx_1
\]
\[
x_1 + x_3 = yx_2
\]
\[
x_2 + x_4 = yx_3
\]
\[
x_3 + x_5 = yx_4
\]
\[
x_4 + x_1 = yx_5,
\]
where \( y \) is a parameter. (Hint: define \( x_0 = x_1, x_7 = x_2 \), etc., and find two different linear recurrences satisfied by \( \{x_n\} \).)

(19) (1967 IMO, no. 6) In a sports contest, there were \( m \) medals awarded on \( n \) successive days \( (n > 1) \). On the first day, one medal and \( \frac{1}{7} \) of the remaining \( m - 1 \) medals were awarded. On the second day, two medals and \( \frac{1}{7} \) of the now remaining medals were awarded; and so on. On the \( n \)-th and last day, the remaining \( n \) medals were awarded. How many days did the contest last, and how many medals were awarded altogether?

(20) (1974 IMO, no. 3) Prove that the number \( \sum_{k=0}^{n} \binom{2n+1}{k+1} 2^{3k} \) is not divisible by 5 for any integer \( n \geq 0 \).

(21) (1980 USAMO, no. 3) Let \( F_r = x^r \sin(rA) + y^r \sin(rB) + z^r \sin(rC) \), where \( x, y, z, A, B, C \) are real and \( A + B + C \) is an integral multiple of \( \pi \). Prove that if \( F_1 = F_2 = 0 \), then \( F_r = 0 \) for all positive integral \( r \).