BERKELEY MATH CIRCLE 2003-2004

Preparation for

Bay Area Mathematical Olympiad 2004

Zvezdelina Stankova

Mills College and UC Berkeley

Problem 1. (Belarus, 1999) There is a 7×7 suqare board divided into 49 unit cells, and tiles of three types: 3×1 rectangles, 3-unit-square corners, and unit squares. Jerry has infinitely many rectangles and one corner, while Tom has only one square.

(a) Prove that Tom can put his square somewhere on the board (covering exactly one unit cell) in such a way that Jerry cannot tile the rest of the board with his tiles.

(b) Now Jerry is given another corner. Prove that no matter where Tom puts his square (covering exactly one unit cell), Jerry can tile the rest of the board with his tiles.

Problem 2. (Iran, 1998) Supposed an $n \times n$ table is filled with the numbers 0, 1, -1 in such a way that every row and column contains exactly one 1 and one -1. Prove that the rows and columns can be reordered so that in the resulting table each number has been replaced by its negative.

Problem 3. (India, 1998) A country has 1998 airports connected by some direct flights. For any three airports, some two are not connected by a direct flight. What is the maximum number of direct flights that can be offered?

Problem 4. (India, 1998) Let n and p be positive integers with $3 \leq p \leq n/2$. Consider a regular n-gon with p vertices colored red and the rest colored blue. Show that there are two congruent nondegenerate polygons each with at least $\lfloor p/2 \rfloor + 1$ vertices, such that the vertices of one polygon are colored red and the vertices of the other polygon are colored blue.

Problem 5. (India, 1998) Let $ABCD$ be a paralelogram. A circle lying inside $ABCD$ touches the lines AB and AD, and intersects BD at E and F. Show that there exists a circle passing through E and F and touching the lines CB and CD.

Problem 6. (India, 1998) Let m and n be natural numbers with $m \geq n \geq 2$. Show that the number of polynomials of degree $2n-1$ with disntinct coefficients from the set $\{1, 2, ..., 2m\}$ which are divisible by $x^{n-1} + \cdots + x + 1$ is

$$2^n n! \left(4\binom{m+1}{n+1} - 3\binom{m}{n} \right).$$