Archimedes

Tom Rike

Berkeley Math Circle January 18, 2004

1 History and Sources

Archimedes lived from 287 BC until he was killed by a Roman soldier in 212 BC. He is usually considered to be one of the three greatest mathematicians of all time, the other two being Newton and Gauss. The extant works of Archimedes are readily available today in Heath [**1**] and Dijksterhuis [**2**]. Both contain the works with extensive notes and historical information. They both can be somewhat difficult to plow through at times. To ease the burden, a new book by Sherman Stein [**3**] came out in 1999 that is accessible to a much broader audience. It fact the only prerequisite is high school algebra and geometry. A partial listing of the works that have not been lost is as follows : On the Equilibrium of Planes, Quadrature of the Parabola, On the Sphere and Cylinder, On Spirals, On Conoids and Spheroids, On Floating Bodies, Measurement of a Circle, The Sand-reckoner, and The Book of Lemmas. In 1906, a palimpsest was found by Heiberg which contained earlier copies of some of Archimedes' work, but more importantly it contained two works which had been lost for 1000 years. One of them disclosed the method that Archimedes employed to arrive at many of his conclusions, the other a treatise *On Floating Bodies* had no surviving copy in Greek. A third item was a short discussion of a puzzle called the Stomachian. Heiberg was not allowed to take the palimpsest, but was able to photograph the pages. From these photographs he made the best translation he could of the document. Soon afterward, the world was at war and the palimpsest disappeared. It surfaced six years ago and was auctioned for $2 million dollars. The annonymous buyer has loaned it to a museum to be restored and studied. Many new discoveries have already been reported. In September, NOVA presented an hour long program about Archimedes and the palimpsest entitled *Infinite Secrets*. [**4**]

2 Calculation of π

Archimedes calculates bounds for π by inscribing and circumscribing polygons of increasing size in a circle. He begins with a hexagon and successively doubles the number of sides until he reaches a 96-gon. He does this without recourse to the decimal system, using very good ratios to approximate the irrationals that appear. Since he only gives the ratios without giving any indication of his method for obtaining them, this has led to a great deal of speculation about the issue. There is also some evidence that this was not his final word on the subject and that he had calculated a more accurate value of π. Knorr[**5**] It can also be shown that using Archimedes' data from the 96-gon computation, the value of π to eighteen decimal places. Phillips [**6**]

In Euclid's *Elements*, Book XII, Proposition 2, following the method of Eudoxus, it is proved that *Circles are to one another as the squares on their diameters*. Proposition 1 in Archimedes, following the method of Eudoxus, shows that the area of a circle is equal to the area of a right triangle with one of the legs equal to the radius and the other equal to the circumference. With Euclid XII.2, this implies the ratio of the circumference of a circle to its diameter is a constant and Archimedes now sets out to find rational bounds for this number. The first thing Archimedes needed was a set of bounds for $\sqrt{3}$. Without explanation Archimedes just writes $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$. In the September issue of *Mathematics Teacher*, I happened to see a letter from a reader, Ken Seidel, of Redwood City which, I believe, provides some insight. He points to an interesting square root agorithm related to continued fractions:

$$\sqrt{a^2+b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \cfrac{b}{2a + \cdots}}}$$

See Heath [**1**] for a discussion of the various methods that historians have put forward. Now Archimedes, using the angle bisector theorem, Euclid VI.3, develops a method of finding the ratio of the side of a $2n$-gon to the diameter of the inscribed circle from the ratio of the side of an n-gon to the diameter. After four interations he multiplies the bounding ratio by 96 to get an upper bound for *pi*. He then develops a method of find the ratio of the side of a $2n$-gon to the diameter of the circumscribing circle from the ratio of the n-gon to the diameter. Again after After four interations he multiplies the bounding ratio by 96 to get an lower bound for π. The final result being that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. Heron reports that Archimedes improved on these results in a lost work *Plinthides and Cylinders*, but the values given are not correct, in fact the lower bound was actually greater than π. The historian Wilbur Knorr put forth the hypothesis that Archimedes would not blunder so greatly, and used the lower bound as an upper bound. The other upper bound which was much too large was found to have a denominator with two digits incorrect, possibly through scribal error. When corrected the the lower and upper bounds reduced by continued fractions yielded 3 + 1/(7 + 1/15) and 3 + 1/(7 + 1/17), approximately 3.141509 and 3.141666. This certainly suggests looking at the continued fraction 3 + 1(7 +1/16) = 355/113 $\approx$ 3.141593. This is the famous ratio used by the Chinese mathematicians in the fifth and sixth centuries. See Knorr[**5**] for more about this.

Some Problems

1. Show that the side of a 96−gon inscribed in a circle with radius one is equal to

$$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$$

2. Let p_n and P_n denote, respectively, half the perimeters of the inscribed and circumscribed polygons with n sides in a circle of radius 1. Use the techniques of Archimedes to show

$$p_{2n}^2 = \frac{2p_n^2}{1+\sqrt{1-p_n^2/n^2}}, \qquad P_{2n} = \frac{2P_n}{1+\sqrt{1+P_n^2/n^2}} \qquad \text{and} \qquad P_n = \frac{p_n}{\sqrt{1-p_n^2/n^2}}.$$

3 The Cattle Problem

From Dijksterhuis [**2**] The cows and bulls of Helios are grazing in the island of Sicily in four herds of different colors: white, black, dappled and yellow. If we call the number of bulls in these herds, respectively, W, Z, P, B and the number of cows similarly w, z,p,b, the following relations between the numbers are given:

$$
\begin{aligned}
W &= (\frac{1}{2}+\frac{1}{3})Z + B \\
Z &= (\frac{1}{4}+\frac{1}{5})P + B \\
P &= (\frac{1}{6}+\frac{1}{7})W + B \\
w &= (\frac{1}{3}+\frac{1}{4})(Z + z) \\
z &= (\frac{1}{4}+\frac{1}{5})(P + p) \\
p &= (\frac{1}{5}+\frac{1}{6})(B + b) \\
b &= (\frac{1}{6}+\frac{1}{7})(W + w)
\end{aligned}
$$

In addition, it required that $W + Z$ be a perfect square, n^2, and $P + B$ be a triangular number, $\frac{m(m+1)}{2}$. For a discussion of the solution to the first seven equations see the first problem in *100 Great Problems of Elementary Mathematics* by Heinrich Dörrie [**7**]. Gabriel Carroll mentioned to me that this book was one of the first mathematics books he read. A "more complete" formulation of the problem was discovered in a Greek manuscript in 1773 in the Wolfenbüttel library. A translation from the Greek to English (via German, since Dörrie's book was originally in German), in poetic form, made up of twenty-two distichs, is also included along with some other historical comments a general discussion of the solution to the complete problem. If you go searching, you will find problem three in this book is a problem on cows and fields by Newton, but it is not nearly as interesting. However, the book is a goldmine of interesting problems in mathematics by the great mathematicians of the past, with very succinctly written solutions.

In Heath [**1**], an alternate interpretation of the bulls forming a "square" came about by considering bulls to be longer than they are wide, and so seeking an answer where the bulls are closely packed to form a "square figure", rather than requiring the number of bulls, $W + Z$, to be a perfect square. This problem is easier and is known as *Wurm's Problem.* It is solved in Heath. Heath then goes on to discuss the solution to the complete problem which leads to the Pellian equation $t^2 - 4729494u^2 = 1$. This type of problem was discussed at the Berkeley Math Circle in the past. For a nice introduction via the discovery method, see the Power Round of the Polya Contest held at Gunn High School[**8**] (October 30, 1999). Unfortunately, Heath also has the incorrect fourth digit for W. This may be where Dijksterhuis got his information, since he has the same error.

In 1889, A.H. Bell, a civil engineer, and two friends formed the Hillsboro, Illinois, Mathematical Club and started the computation of the solution to the complete problem. After four years they computed the first 32 left-hand digits and the last 12 right-hand digits for each variety of bulls and cows, as well as the total number of cattle in the herd. This is detailed in Albert H. Beiler's *Recreations in the Theory of Numbers* [**9**]. However, there is a misprint in the book. The value printed for the variable t is actually the value of t^2. A truncated version of the problem in prose is also given. The discussion of the problem appears in the chapter entitled *The Pellian* where you will find out why Pell, who had almost nothing to do with solving this type of equation has his name gloriously attached to it. There is also a fairly clear presentation of the method of solution via continued fractions.

The first complete listing of the solutions to the problem was given in 1965 by the Canadian mathematicians H.C. Williams, R.A. German, and C.R. Zarnke, who computed it using a computer. The computer printout is on deposit among Unpublished Mathematical Tables at the University of Maryland. This showed that the last two of Bell's thirty-two left-hand digits were incorrect. In 1980, Harry L. Nelson of the Lawrence Livermore National Laboratory recast the problem in code suitable for exercising the newly delivered CRAY-1 computer. The computation of the solution, together with extensive checking was done in ten minutes. Since this was not of sufficient length for the purpose desired, the code went on to find five additional solutions, the largest of which has well over a million digits. All 206,545 digits of the smallest solution taking up over 46 computer pages (64 rows of 70 digits) are printed at one-third actual size, four-to-a-page, in the *Journal of Recreational Mathematics* [**10**]. See the recent article by Vardi [**11**] for a beautiful discussion of personal computer techniques and a prose translation of the problem.

4 References

1. T.L. Heath. *The Works of Archimedes* .Dover Pubilications,Inc. Reissue of the 1897 edition with the 1912 Supplement on *The Method*.

2. E. J. Dijksterhuis. *Archimedes*. Princeton University Press, 1987.

3. Sherman Stein. *Archimedes: What Did He Do Besides Cry Eureka*. MAA, 1999.

4. NOVA. *Infinite Secrets*. http://www.pbs.org/wgbh/nova/archimedes/palimpsest.html

5. Wilbur Richard Knorr. *The Ancient Tradition of Geometric Problems*. Dover, 1993.

6. George M. Phillips. *Two Millenia of Mathematics*. Canadian Mathematical Society Springer-Verlag New York Inc., 2000.

7. Heinrich Dörrie. *100 Great Problems of Elementary Mathematics*. Dover Publications, Inc.,1965.

8. Joshua Zucker. *Seventh Polya Contest*. http://www.castilleja.org/faculty/josh_zucker/polya/9900/index.html

9. Albert H. Beiler. *Recreations in the Theory of Numbers*. Dover Publications,Inc., 1964.

10. Harry L. Nelson. *Journal of Recreational Mathematics*, pp 162-176, Vol. 13 No. 3, 1980-81.

11. Ilan Vardi. Archimedes Cattle. *The American Mathematical Monthly*, pp 305-319 Vol. 105 No. 4, April 1998. Mathematical Association of America.

12. Fan Chung and Ron Graham. *Tour of Archimedes' Stomach*. http://math.ucsd.edu/ fan/stomach/

If you have comments, questions or find glaring errors, please contact me by e-mail at the following address: tricycle222@earthlink.net