COMBINATORICS AND GEOMETRY
BERKELEY MATH CIRCLE

Combinations.

What is combinatorics? It is a way of counting some combinations. What is a combination? It will become clear from the problems below.

1. Helen has 15 skirts, 12 shirts and 3 pairs of shoes. How many ways can Helen dress for tonight party (provided she likes her shoes matched)?

Solution. Helen is a lucky girl. She has 15 ways to choose a skirt; for each choice of a skirt she has 12 ways to choose a shirt. This is already $15 \times 12 = 180$ choices. To each of this outfit she has 3 choices for a pair of shoes. Hence the total number of choices is $180 \times 3 = 540$. Thus, Helen can have a new outfit for one year and a half!

This simple problem teaches us one idea: to multiply possibilities. Using this idea, one can do more complicated problems.

2. How many ways are there to put one white and one black knight on a chess board so that they do not attack each other?

3. There are 10 houses on one side of a street, and you have three paints (green, yellow and red) to paint them. How many ways are there to paint the houses?

4. The same question as in the previous problem but the city council requires that neighboring houses have different colors.

5. How many diagonals does a convex n-gon have?

6. How many 5-letter "words" can one make out of 26 letters, under the condition that a "word" has at least one vowel?

Permutations.

7. 6 horses participate in a race. How many ways are there to finish the race if we assume that no two horses finish simultaneously?

Solution. The last problem is asking how many ways are there to arrange 6 numbers in a row. There are 6 choices for a winner, 5 choices for the second horse, etc. The total number of ways is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Ask the same question about n horses where n is any natural number; the answer is given by the number $n! = n\,(n-1)\,(n-2)\ldots 1$. Each ordering of numbers from 1 to n is called a permutation. Hence $n!$ is the number of permutations of n elements.

8. There are n boys and n girls in a dance class. How many ways are there to divide them in pairs for a tango?

Date: November 28, 2006 .

Binomial coefficients.

9. A script-writer decided that in his plot, James Bond must have 10 girlfriends, Alice, Barbara, Clara, Denise, Eleanor, Fiona, Georgina, Hannah, Irene, and Janelle; exactly 3 of which are spies. How many different combinations of spies can there be?

Solution. Write the names of spies. There are 10 choices for the first name, 9 for the second and 8 for the third; therefore there are $10 \times 9 \times 8 = 720$ possible lists. Since the order of names in the list is not important we need to divide by $3! = 6$ ways to reorder the names. So, there are 120 possibilities.

Generalize the previous problem, ask how many ways are there to choose k objects from a collection of n objects. This number is denoted by $\binom{n}{k}$ (read: n choose k) and is given by the formula

$$\binom{n}{k} = \frac{n\,(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}.$$

10. Jack has three apples, two oranges and four bananas. Jack's mother wants to write a menu for the next nine days, one fruit per day. How many ways are there to do this?

11. Prove the identities

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \ \binom{n}{k} = \binom{n}{n-k}$$

without using of the formula for $\binom{n}{k}$. Use the combinatorial definition of $\binom{n}{k}$.

12. Prove the identity

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

13. Show that

$$\binom{n}{0} - \binom{n}{1} + \dots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0.$$

14. Show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}.$$

Pascal's triangle.

15. Consider a rectangular grid. How many ways are there to go from point (0,0) to the point (m, n) if you can only move up or to the right along the grid lines?

At each point (m, n) write the number of paths from (0,0) to this point; you obtain the Pascal triangle. Usually, one draws the Pascal triangle in a slightly different manner; rotating it so that point (0,0) is on top.

16. Use the Pascal triangle to calculate

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^k \binom{n}{k},$$

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k}.$$

Sometimes binomial coefficients appear in disguise.

17. How many ways are there to divide n identical oranges between k people?

18. The same question, but everybody gets at least one orange.

19. A train with m passengers makes n stops. How many ways are there for these passengers to get off the train?

20. The same question, but we only need to know the number of passengers exiting at each stop.

Recursion and Fibonacci numbers.

A sequence $f_1, f_2, f_3, \ldots$ satisfying

$$f_1 = f_2 = 1,\ f_n = f_{n-1} + f_{n-2}$$

is called the Fibonacci sequence and its terms are called Fibonacci numbers. Let us see how Fibonacci numbers appear in combinatorics.

21. There are n steps from the ground floor to Jenn's apartment. She can go by one step, by two steps or in any combination. How many ways are there to reach her apartment?

22. Find the number of all subsets of the set $\{1, 2, \ldots, n\}$ which do not contain two consecutive numbers.

23. How many ways are there to write n as a sum $n = a_1 + a_2 + \cdots + a_k$ such that each a_i is an odd natural number?

In three problems above we exploit the idea of recursion: express unknown $f(n)$ in terms of $f(k)$ for $k < n$.

Inclusion-exclusion principle.

24. There are 30 students in the eight grade, 10 are in math club, 6 are in chess club, and 11 are on the volleyball team. Only one student participates in all three activities. 3 students from the volleyball team are also in the chess club; 3 students from the chess club are in the math club; 4 students from the math club are on the volleyball team. How many students do not participate in any of these three activities?

25. Answer the same question as in Problem 4 except the houses are not in a row but around a circle.

26. A lord had 10 guests for dinner. Each guest gave his hat to a valet. The lord is a practical joker. After the dinner he asked his valet to give the hats back so that none of the guests gets his own hat. How many ways are there to do it?

Geometry.

27. n lines on a plane cut the plane into parts. Assume that every two lines intersect and there is no triple intersections. Find the number of parts. How many of these parts are bounded?

28. Consider n distinct lines on the plane without any restriction on how they intersect. Let P be a point of intersection. Define the index $i(P)$ to be the number of lines passing through P minus 1. Show that the number of parts in which the lines cut the plane equals $n+1$ plus the sum of indexes of all points. Find the the number of bounded parts.

29. Given n planes in a space, any two planes meet in a line, any three planes meet at a point, any four planes do not have common points. Find the number of parts in which the planes divide the space. Find the number of bounded parts.

30. Answer the questions of the previous problem without any assumption on a position of the planes (in terms similar to the Problem 28).

31. A triangulation of a convex n-gon is a way to cut it into triangles by drawing some diagonals which do not meet inside the n-gon. Prove that the number of triangles does not depend on a choice of diagonals.

32. Let c_n denote the number of triangulations of an $n+2$-gon, we assume $c_0 = 1$. So $c_1 = 1$, $c_2 = 2$, $c_3 = 5$. Prove that c_n satisfies the relation

$$c_n = c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-1} c_0.$$

The numbers c_n are called the Catalan numbers.

33. Show that c_n counts the number of paths from (0,0) to (n, n), which never go above the line $y = x$. Paths are defined in the same way as in Problem 15.

34. Show that c_n counts the number of words which contain n letters a and n letters b such that after erasing any number of letters at the end the number of remaining $a's$ is not less than the number of remaining $b's$.

35. Prove that $c_n = \frac{1}{n+1}\binom{2n}{n}$.