BERKELEY MATH CIRCLE 2001–2002

Practice Exam I for BAMO 2002
Greek Mathematical Competitions 2001

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1. 18^{TH} Hellenic Mathematical Olympiad 2001 "ARCHIMEDES"

(1) The integers from 1 to 500 are written on a blackboard. Two students A and B play the following game: the students alternate deleting each one number. The ends when only two numbers are left. The winner is B if the sum of the two numbers is divisible by 3; otherwise the winner is A. If A starts, does student B have a winning strategy?

(2) Prove that there are no $\alpha, \beta \in \mathbb{N}$, and $k \in \mathbb{Z}$ such that $(15\alpha + \beta)(\alpha + 15\beta) = 3^k$.

(3) Let $\triangle ABC$ be inscribed in a circle k of radius R. BL and CM are the angle bisectors of $\angle B$ and $\angle C$, respectively, and line LM intersects the arc AB not including C in K. We draw $KA_1 \perp BC$, $KB_1 \perp AC$ and $KC_1 \perp AB$, where $A_1 \in BC$, $B_1 \in AC$ and $C_1 \in AB$. Let x be the distance from L to the sides BA, BC and y be the distance from M to the sides CA, CB. Prove that $\dfrac{1}{KB} = \dfrac{1}{KA} + \dfrac{1}{KC}$.

(4) Let $f : \mathbb{N} \to \mathbb{R}$ be a function such that $f(1) = 3$ and

$$f(m+n) + f(m-n) - m + n - 1 = \frac{f(2m) + f(2n)}{2},$$

for all non-negative integers m, n with $m \geq n$. Find f.

2. Selection Examination for Junior Balcan Math Olympiad 2001

(1) Four men are standing at the entrance to a dark tunnel. Man A needs 10 minutes to cover the length of the tunnel, B needs 5 minutes, C needs 2 minutes and D needs 1 minute. They have only 1 torch which must be used by anyone crossing the tunnel. Moreover, at most two men can cross at the same time the tunnel using the torch. Find the shortest possible time needed for the four men to go through the tunnel.

(2) Prove that there are no integers x, y, z which satisfy the equation

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = 2000.$$

(3) Let $ABCD$ be a quadrilateral with $\angle DAB = 60^\circ$, $\angle ABC = 90^\circ$ and $\angle BCD = 120^\circ$. The diagonals AC and BD intersect in M, and $BM = \alpha$, $MD = 2\alpha$. Let O be the midpoint of AC. Draw $OH \perp BD$ with $H \in BD$, and $MN \perp OB$ with $N \in OB$. Find the area of $ABCD$.

(4) Find all positive integers N which are perfect cubes not divisible by 10, such that dropping their last three digits results in another perfect cube.

3. Selection Examination for the International Math Olympiad 2001

(1) Let k be a circle, and P, Q – points on k. Let M be the midpoint of PQ, and A and C – points on k such that AC passes through M. $ABCD$ is a trapezoid with k as its circumcircle and $AB||CD||PQ$. Prove that AD and BC intersect in a point X independent of the choice of A on k.

(2) Find all integers n for which the polynomial $p(x) = x^5 - nx - n - 2$ can be written as a product of two non–constant polynomials with integer coefficients.

(3) Prove that there exists a positive integer N such that the decimal representation of 200^N starts with the sequence of digits 200120012001.

(4) An equilateral $\triangle ABC$ of side 1 is given, and S denotes the set of all points lying in its interior or on its boundary. For any M in S, $a(M)$, $b(M)$ and $c(M)$ denote the distances from M to the sides BC, CA and AB, respectively. Let

$$f(M) = a(M)^3\left(b(M) - c(M)\right) + b(M)^3\left(c(M) - a(M)\right) + c(M)^3\left(a(M) - b(M)\right).$$

(a) Describe geometrically the set of points M in S for which $f(M) \geq 0$.

(b) Find the maximal and the minimal values of $f(M)$ when M moves inside S, and find the points where these values are attained.

4. 18^{TH} Balcan Mathematical Olympiad 2001

(1) Let n be a positive integer. Show that if a and b are integers greater than 1 with $2^n - 1 = ab$, then the number $ab - (a - b) - 1$ is of the form $k \cdot 2^{2m}$, where k is odd and m is a positive integer.

(2) Prove that for a convex pentagon all its interior angles are congruent and the lengths of its sides are rational numbers, then the pentagon is regular.

(3) Let $a, b, c > 0$ such that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq abc\sqrt{3}$.

(4) A cube of dimensions $3 \times 3 \times 3$ is divided into 27 congruent unit cubical cells. One of these cells is empty and the others are filled with unit cubes labelled in an arbitrary manner with the numbers 1,2,...,26. An *admissible move* is the moving of a unit cube into an adjacent empty cell. Is there a finite sequence of admissible moves after which the unit cube labeled with k and the unit cube labelled with $27 - k$ are interchanged, for each $k = 1, 2, ..., 13$? (Two cells are said to be adjacent if they share a common face.)