BAMO Practice Session

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1 Plan for the day.

The 2001 Bay Area Mathematical Olympiad is on Tuesday, February 27. I want to briefly discuss two vague questions: What can you do in the next two weeks to increase your chances of doing well on this contest? What can you do *during* the contest to maximize your score?

Then, let's walk through a few problems, thinking about (A) how to solve them – that is, how do you come up with the insight you need to write a solution and (B) how do you write the solution effectively.

2 One or two topics you might do well to expect

It's probably foolish to try to predict what topics will come up on this year's BAMO – there are too many potential topics and only a few questions. It's still worth studying the problems from the two years of BAMOs, and from other similar olympiads, to get a sense of what sort of questions are likely to be on it.

2.0.1 Inequalities

Many Olympiads have one "inequality" problem every year. (Look at the Canadian Olympiads included here, or look at the British Mathematical Olympiads. The BAMO had one in 2000, but not in 1999.)

I'm not going to do a full presentation on inequalities – that has been done in the math circle before! (see http://mathcircle.berkeley.edu/inequalities.pdf for a copy of Bjorn Poonen's handout from last year), but some of these are very commonly used. You can generally get away with citing any of these without having to reprove them, so long as you use them authoritatively. Rather than give the formulas here, I'm just going to describe them qualitatively.

- Arithmetic Mean – Geometric Mean inequality For any set of non-negative numbers, the arithmetic mean is greater than or equal to the geometric mean, with equality only when all the terms are equal. You can toss the Harmonic Mean and the Quadratic Mean in here, too.
- Cauchy-Schwarz The absolute value of the dot product of any two vectors is less than or equal to the product of the lengths of the vectors, with equality only when the vectors are linearly dependent.
- Hölder If p and q are positive real numbers for which $1/p + 1/q = 1$, then the dot product of any two vectors is less than or equal to the product

of the p-norm of one vector and the q-norm of the other. (Note when $p = q = 2$, this is the Cauchy-Schwarz inequality) (Also note the definition of a p-norm).

- Jensen For any convex function, the arithmetic mean of a set of values of the function image taken on a set of input values is greater than or equal to the value of the function taken at the mean of those input values. (actually almost all of the above inequalities can be interpreted as instances of Jensen's inequality!)
- Re-arrangements and Chebychev If you have the dot product of two vectors (with non-negative components), and you can rearrange the order of the components of one of the vectors, you will maximize the dot product by being sure the largest components of the two vectors are multiplied together, followed by the next largest, etc. – you'll minimize the product by being sure the largest of one is paired with the smallest of the other, the second largest with the second smallest, etc. Chebychev inequality comes from this.
- Other? Geometric inequalities?

Sometimes inequalities are phrased in terms of "find the maximum" or "find the minimum". Many similar techniques apply here. This leads into extremal problems ...

2.1 Invariants, Monovariants

There has been a lot of attention to these concepts already during the math circle, and with good reason. It is definitely true that many olympiad-style problems involve invariants or monovariants. See http://mathcircle.berkeley.edu/BMC3/monovar.pdf for Gabriel Carroll's handout from this year with many good problems in it. Incidentally, that handout includes a proof of the rearrangement inequality above.

Which problems from the last two years of BAMOs actually use invariants or monovariants in their solutions?

2.2 Other topics?

Well, there's always at least one problem involving some geometric concept, but I quailed before trying to come up with a brief summary of these concept.

Obviously, there will be some problems that require at least elementary combinatorics and probably there will be one requiring elementary number thoery.

3 Various Problems

3.1 (Iran, 1999) Does there exist a positive integer which is a power of 2 from which we can obtain another power of 2 by rearranging its digits?

3.2 (Iran, 1999) Consider the triangle ABC with the angles B and C each larger than 45 degrees. Construct right isosceles triangles CAM and BAN outside the triangle ABC with right angles ∠CAM and ∠BAN and right isosceles triangle BPC inside ABC with right angle ∠ BPC . Prove that the triangle MNP is also right isosceles.

3.3 (Iran, 1999) We have a 100×100 lattice with a tree on each of the 10000 points. (The points are equally spaced.) Find the maximum number of trees we can cut such that if we stand on any cut tree, we see no trees which have been cut. (In other words, on the line connecting any two trees that have been cut, there should be at least one tree which hasn't been cut.)

3.4 (Iran, 1999) Find all natural numbers m such that :

$$m = 1/a_1 + 2/a_2 + 3/a_3 + ... + 1378/a_{1378}$$

where $a_1, \ldots, a_{1378}$ are natural numbers.

3.5 (Iran, 1999) Consider the triangle ABC. P, Q, and R are points on the sides AB, AC, and BC respectively. A' , B' and C' are points on the lines PQ , PR and QR respectively such that AB is parallel to A'B', AC is parallel to A'C', and BC is parallel to B'C'. Prove that AB/A'B' = $S_{\text{ABC}}/S_{\text{PQR}}$ (S_{ABC} means the area of the figure ABC)

3.6 (Iran, 1999) $A_1, A_2, \ldots, A_n$ are n distinct points on the plane. We color the middle of each line A_iA_j $(i \neq j)$ red. Find the minimum number of red points.

3.7 (Putnam 1988) Show that every (positive) composite integer is expressible as $xy + xz + yz + 1$ with x, y, and z positive integers.

3.8 (Putnam 1986) What is the units digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

3.9 (Bulgaria 1980) Prove that for every 3 non-negative integers a, b, and c, the inequality

$$a^3 + b^3 + c^3 + 6abc \geq \frac{(a+b+c)^3}{4}$$

holds, with equality only when two of these numbers are equal and the third is zero.

3.10 (Bulgaria 1981) Prove that if n is a positive integer for which the number $1 + 2^n + 4^n$ is prime, then n is a power of 3.

3.11 (Bulgaria 1985) Let $P(x)$ be a non-constant polynomial with integer coefficients. For any positive integers n and k, show that there are n consecutive positive integers a,$a+1$,..., $a+n-1$ such that each of $P(a)$, $P(a+1)$, $\ldots P(a+n-1)$ has at least k prime divisors.

3.12 (British Math Olympiad 1991) Prove that the number $3^n+2\times 17^n$, where n is a non-negative integer, is never a perfect square.

3.13 (British Math Olympiad 1991) Find all positive integers k such that the polynoimal $x^{2k+1} + x + 1$ is divisible by the polynomial $x^k + x + 1$. For each such k, specify integers n for which $x^n + x + 1$ is divisible by $x^k + x + 1$.

3.14 (British Math Olympiad 1991) The quadrilateral $ABCD$ is inscribed in a cricle of radius r. The diagonals AC and BD meet at E. Prove that if AC is perpendicular to BD, then $EA^2 + EB^2 + EC^2 + ED^2 = 4r^2$. Is it true that if this equation holds, then AC is perpendicular to BD? justify your answer.

3.15 (British Math Olympiad 1991) Find, with proof, the minimum value of $(x+y)(y+z)$ where x, y, and z are positive real numbers satisfying the condition $xyz(x + y + z) = 1$.

3.16 (British Math Olympiad 1991) Find the number of permutations $j_1, j_2, j_3, j_4, j_5, j_6$ of 1,2,3,4,5,6 with the property that, for no integer n, $1 \leq n \leq 5$, do $j_1, j_2, \ldots, j_n$ form a permutation of $1, 2, = ldots, n$.

3.17 (British Math Olympiad 1991) Show that if x and y are positive integers such that $x^2 + y^2 - x$ is divisible by $2xy$, then x is a perfect square.

3.18 (British Math Olympiad 1991) A ladder of length l rests against a vertcial wall. Suppose that there is a rung on the ladder which has the same distance d from both the wall and the (horizontal) ground. Find *explicitly*, in terms of l and d, the height from the ground that the ladder reaches up the wall.

3.19 (Canada, 2000) At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners.)

3.20 (Canada, 2000) A *permutation* of the integers $1901, 1902, \ldots, 2000$ is a sequence $a_1, a_2, \ldots, a_{100}$ in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

$$s_1 = a_1, \ \ s_2 = a_1 + a_2, \ \ s_3 = a_1 + a_2 + a_3, \ \ldots \ , \ s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

How many of these permutations will have no terms of the sequence $s_1, \ldots, s_{100}$ divisible by three?

3.21 (Canada, 2000) Let $A = (a_1, a_2, \ldots, a_{2000})$ be a sequence of integers each lying in the interval $[-1000, 1000]$. Suppose that the entries in A sum to 1. Show that some nonempty subsequence of A sums to zero.

3.22 (Canada, 2000) Let $ABCD$ be a convex quadrilateral with

$$\begin{aligned} \angle CBD &= 2\angle ADB, \\ \angle ABD &= 2\angle CDB \\ \text{and} \qquad AB &= CB. \end{aligned}$$

Prove that $AD = CD$.

3.23 (Canada, 2000) Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$\begin{aligned} a_1 \geq a_2 \geq \cdots \geq a_{100} &\geq 0, \\ a_1 + a_2 &\leq 100 \\ \text{and} \qquad a_3 + a_4 + \cdots + a_{100} &\leq 100. \end{aligned}$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.

3.24 (Canada, 1999) Find all real solutions to the equation $4x^2 - 40[x] + 51 = 0$.
Here, if x is a real number, then $[x]$ denotes the greatest integer that is less than or equal to x.

3.25 (Canada, 1999) Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and center on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

3.26 (Canada, 1999) Determine all positive integers n with the property that $n = (d(n))^2$. Here $d(n)$ denotes the number of positive divisors of n.

3.27 (Canada, 1999) Suppose $a_1, a_2, \ldots, a_8$ are eight distinct integers from $\{1, 2, \ldots, 16, 17\}$. Show that there is an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Also, find a specific set of 7 distinct integers from $\{1, 2, \ldots, 16, 17\}$ such that the equation $a_i - a_j = k$ does not have three distinct solutions for any $k > 0$.

3.28 (Canada, 1999) Let x, y, and z be non-negative real numbers satisfying $x + y + z = 1$. Show that

$$x^2y + y^2z + z^2x \leq \frac{4}{27},$$

and find when equality occurs.

3.29 (Canada, 1998) Determine the number of real solutions a to the equation

$$\left[\frac{1}{2}\,a\right] + \left[\frac{1}{3}\,a\right] + \left[\frac{1}{5}\,a\right] = a\ .$$

Here, if x is a real number, then $[\,x\,]$ denotes the greatest integer that is less than or equal to x.

3.30 (Canada, 1998) Find all real numbers x such that

$$x = \left(x - \frac{1}{x}\right)^{1/2} + \left(1 - \frac{1}{x}\right)^{1/2}.$$

3.31 (Canada, 1998) Let n be a natural number such that $n \geq 2$. Show that

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

3.32 (Canada, 1998) Let ABC be a triangle with $\angle BAC = 40^\circ$ and $\angle ABC = 60^\circ$. Let D and E be the points lying on the sides AC and AB, respectively, such that $\angle CBD = 40^\circ$ and $\angle BCE = 70^\circ$. Let F be the point of intersection of the lines BD and CE. Show that the line AF is perpendicular to the line BC.

3.33 (Canada, 1998) Let m be a positive integer. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 0$, $a_1 = m$, and $a_{n+1} = m^2 a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$. Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

3.34 (AMM, 1989, Anderson, Lovasz, Shor et al) A game is played with n pebbles. At the start they are all in a pile at position 0 (the positions may be thought of as integers on a number line). On every turn of the game, each pile on the board is simultaneously divided into two equal subpiles (with one pebble left over if the number of pebbles in the pile is odd). One subpile is moved one position to the left, one is moved one position to the right, and if one pebble is left over, it stays in its original position. The game ends when (and if) all non-empty piles have one pebble. So for example, one play of the game might be:

at start:			5		
after one turn:		2	1	2	
after two turns:	1	0	3	0	1
after three turns:	1	1	1	1	1

Prove that, for any odd n, the game will eventually end with a row of n consecutive piles of one pebble each.