Algebraic Geometry

Question: What regular polygons can be inscribed in an ellipse?

1. Varieties, Ideals, Nullstellensatz

Let K be a field. We shall work *over* K, meaning, our coefficients of polynomials and other scalars will lie in K.

Definitions.

1. An *affine variety* X in $\mathbb{A}^n$ is the zero locus of a collection of polynomials $\{f(x_1, ..., x_n)\}$ in $K[x_1, ..., x_n]$. A *projective variety* X in $\mathbb{P}^n$ is the zero locus of a collection of *homogeneous* polynomials $\{F(Z_0, ..., Z_n)\}$ in $K[Z_0, ..., Z_n]$.
2. For a given variety X, the set of *all* polynomials vanishing on X is an ideal, called the *ideal* of X and denoted by $I(X)$. In other words,
$$I(X) = (..., f_\alpha, ...) \text{ or } I(X) = (..., F_\alpha, ...),$$
depending on whether the variety is affine or projective. In the second case, the ideal is called *homogeneous*, i.e. generated by homogeneous polynomials. Conversely, for a given ideal $I \subset K[x_1, ..., x_n]$ (or homogeneous $I \subset K[Z_0, ..., Z_n]$), the zero locus of I is denoted by $Z(I)$.

Projective varieties can be thought of as "completions", "compactifications", or "closures" of affine varieties. Their global properties are usually easier to describe than those of affine varieties. Conversely, affine varieties can be thought of as building blocks of projective varieties (indeed, they constitute an open cover), and hence local properties are easier to describe using affine varieties. However, projective varieties vary "nicely" in families and hence parametrizing and moduli spaces are usually constructed for projective varieties with certain defining common properties.

3. A ring R is called *Noetherian* if any inscreasing sequence of ideals terminates, i.e. whenever I_j's are ideals in R such that
$$I_1 \subset I_2 \subset \cdots \subset I_j \subset \cdots$$
then for some $k \geq 1$: $I_k = I_{k+1} = I_{k+2} = \cdots$
4. An ideal I in a ring R is called *radical* if whenever $f^m \in I$ ($f \in R$, $m \in \mathbb{N}$), then $f \in I$. In other words, I contains all (positive integer) roots of its elements.

Theorem 1. *The polynomial ring $R = K[x_1, x_2, ..., x_n]$ is Noetherian. Consequently, any ideal I of R is finitely generated. In particular, for any affine (or projective) variety X, the ideal I of X is generated by finitely many polynomials.*

Lemma 1. *The ideal of any algebraic variety X is radical. If I is an arbitrary ideal, the set of all radicals of its elements:*

$$\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N}\}$$

is also an ideal, called the radical of I. If I is a radical ideal, then its radical is itself, i.e. the operation of taking radicals stabilizes after one step.

We construct maps bewteen the set $\mathcal{X}$ of all varieties X over K and the set $\mathcal{J}$ of all ideals $J \subset K[x_1, ..., x_n]$ by sending

$$i : X \mapsto I(X) \text{ and } j : J \mapsto Z(J).$$

It is immediate from definition of an ideal of variety that $Z(I(X)) = X$, i.e. $j \circ i = \mathrm{id}_{\mathcal{X}}$. Also, the image of i is inside $\mathcal{R}$, the subset of radical ideals in R. Thus, we have an injection $i : \mathcal{X} \hookrightarrow \mathcal{R}$ with a one–way inverse $j : \mathcal{R} \to \mathcal{X}$. It is natural to ask whether i and j are inverses of each other, i.e. whether $i \circ j = \mathrm{id}_{\mathcal{R}}$.

For any ideal $J \subset \mathcal{J}$ (not necessarily radical), we consider $X := Z(J) = j(J)$ – a variety, and then take $i(X) = I(X) = I(Z(J))$. It is evident that $I(X)$ will be a radical ideal containing J, but is it going to be $\sqrt{J}$? To paraphrase the problem, start with J being a radical ideal and take $I(Z(J))$. Is this equal to J?

The answer in general is *no*. For example, if $K = \mathbb{R}$ is the ground field, and $J = (x^2 + y^2)$ is the ideal generated by the single polynomial $f(x, y) = x^2 + y^2$ in the affine plane, then J is obviously radical (f is irreducible), and the zero locus of J is $Z(J) = (0, 0)$ – just one point. However, the ideal of $(0, 0)$ is definitely much larger than J – it consists of all polynomials vanishing at $(0, 0)$, i.e. having no free terms: $I((0, 0)) = (x, y) \supset (x^2 + y^2)$. Thus, we end up with a (radical) ideal bigger than the original.

The above situation is possible because $\mathbb{R}$ is not an algebraically closed field. This leads to the famous *Nullstelensatz*, a basic theorem in commutative algebra, on which much of algebraic geometry over algebraically closed fields is based.

Theorem 2. *(Nullstelensatz) If K is an algebraically closed field, then for any ideal $J \subset K[x_1, ..., x_n]$:*

$$i \circ j(J) = I(Z(J)) = \sqrt{J}.$$

In particular, there is a one–to–one correspondence between the set $\mathcal{X}$ of affine varieties X in $\mathbb{A}^n$ and radical ideals $\mathcal{R}$ given by

$$i : \mathcal{X} \to \mathcal{R}, \text{ and } j : \mathcal{R} \to \mathcal{X}.$$

Note that the radical of the unit ideal is again the unit ideal: $\sqrt{(1)} = (1)$. This implies the following corollary:

Corollary 1. *If $f_1, ..., f_k$ are polynomials in several variables over an algebraically closed field K, then they have no common zeros in K iff*

$$1 = g_1 f_1 + g_2 f_2 + \cdots + g_k f_k$$

for some polynomials $g_1, g_2, ..., g_k$.

Some other "strange" things happen over fields, which are not algebraically closed. For example, we would like to call a "planar curve" any variety X in $\mathbb{A}^2$ which is given by 1 polynomial. However, over $\mathbb{R}$, the "curve" defined by $x^2 + y^2 = 0$ is really just a point, while over $\mathbb{C}$ (or any algebraically closed fields) it is a pair of intersecting lines. Thus, many interesting and intuitive properties of algebraic varieties hold only over algebraically closed fields.

There is an analog of Nullstelensatz for projective varieties (for $K = \overline{K}$, of course.) There is one subtle point, though. We call $Ir = (Z_0, ..., Z_n)$ the *irrelevant ideal* in $K[Z_0, Z_1, ..., Z_n]$. Note that Ir is radical, and that $Z(Ir) = \emptyset$. Yet, Ir is *not* the whole ideal of $\emptyset$: $(1) = K[Z_0, Z_1, ..., Z_n]$, the unit ideal, is the ideal of $\emptyset$. Thus, we have two radical ideals competing for the $\emptyset$: $Z(Ir) = Z((1)) = \emptyset$. The bigger one "wins", because $I(\emptyset) = (1)$, and we state the Nullstelensatz as follows:

Theorem 3. *There is a one–to–one correspondence between the set* $\mathfrak{X}$ *of projective varieties* $X \subset \mathbb{P}^n$ *and the set* $\mathfrak{R}$ *of radical homogeneous ideals minus* Ir *given by* i *and* j *from above. In particular, for* $J \in \mathfrak{J}$*:*

$$I(Z(J)) = \begin{cases} \sqrt{J} & \text{if } Z(J) \neq \emptyset \\ (1) & \text{if } Z(J) = \emptyset. \end{cases}$$

Note further that for a (homogeneous) ideal J, $Z(J) = \emptyset$ iff $\sqrt{J} = (1)$ or $\sqrt{J} = (Z_0, Z_1, ..., Z_n)$. In both cases, $\sqrt{J} \supset (Z_0, Z_1, ..., Z_n)$, which can be shown to imply $J \supset (Z_0, Z_1, ..., Z_n)^d$ for some $d > 0$.

Proposition 1. *Let* J *be a homogeneous ideal. Then* $Z(J) = \emptyset$ *iff* J *contains a power of the irrelevant ideal. In other words, a collection of homogeneous polynomials* $\{F_\alpha\}$ *will have no common zeros iff the ideal generated by the* F_α*'s contains all (homogeneous) polynomials of a certain degree* $d > 0$.

Example 1. Let Q be the (smooth) quadric surface in $\mathbb{P}^3$ given as the zero locus of one homogeneous quadratic equation:

$$Q = Z(Z_0 Z_3 - Z_1 Z_2).$$

The quadric Q consists of two families of lines, each of which sweeps Q on its own:

$$\{Z_1 = \lambda Z_0,\ Z_3 = \lambda Z_2\} \text{ and } \{Z_2 = \mu Z_0,\ Z_3 = \mu Z_1\}.$$

In terms of the matrix

$$M = \begin{pmatrix} Z_0 & Z_1 \\ Z_2 & Z_3 \end{pmatrix},$$

Q is the locus where $\det M = 0$, one family of lines consists of lines where the two column satisfy a given linear relation, the other family – where the two rows satisfy a given linear relation. Note that two lines in a family do not intersect (they are skew lines in $\mathbb{P}^3$), while any two lines from different families intersect in exactly one point. The latter hints at an alternative description of Q – namely as the "Cartesian product" $\mathbb{P}^1 \times \mathbb{P}^1$.

More generally, for any two varieties X and Y, there is a (unique) variety $X \times Y$ with projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$, the fibers of which are correspondingly copies of Y and of X. The uniqueness of such a variety is ensured by some extra (natural) properties coming from category theory – such properties are exactly what we would expect from a variety deserving to be called the "product" of two varieties.

The actual construction of this product is given by a (seemingly random) map, called the *Segre embedding*. For starters, to construct the product $\mathbb{P}^n \times \mathbb{P}^m$, we define the Segre map by

$$\sigma_{n,m} : \mathbb{P}^n_{[X]} \times \mathbb{P}^m_{[Y]} \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}_{[Z]}$$

$$\sigma_{n,m}([X_0, ..., X_n], [Y_0, ..., Y_m]) = [X_0Y_0, ..., X_iY_j, ..., X_nY_m].$$

If we label the coordinates on $\mathbb{P}^{(n+1)(m+1)-1}_{[Z]} = \mathbb{P}^N$ by Z_{ij} for $0 \leq i \leq n$ and $0 \leq j \leq m$, then the image of $\sigma_{n,m}$ is the zero locus of all possible quadratic equations $Z_{ij}Z_{kl} = Z_{il}Z_{kj}$. Thus, the image of the Segre map is a variety. Factoring into this the injectivity of $\sigma_{n,m}$, allows us to "identify" the *set*–product $\mathbb{P}^n \times \mathbb{P}^m$ with a subvariety of $\mathbb{P}^N$, which is *define* as the *product* of $\mathbb{P}^n$ and $\mathbb{P}^m$ in the *category of varieties*. One can show that this particular variety satisfies all the required properties of a product, and by virtue of the *uniqueness* of the product, it is *The product variety* $\mathbb{P}^n \times \mathbb{P}^m$.

Now it is not hard to construct the product of any two varieties X and Y: if $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$, then $X \times Y$ is the subvariety of $\mathbb{P}^n \times \mathbb{P}^m$, satisfying the extra equations coming from X and Y.

Example 2. The *twisted cubic* curve C is the zero locus of three polynomials:

$$C = Z(Z_0Z_3 - Z_1Z_2, Z_0Z_2 - Z_1^2, Z_1Z_3 - Z_2^2).$$

The twisted cubic C lies on the quadric surface Q, and is a curve of type $(2,1)$ on Q, i.e. C meets every line in one family in 2 points, and every line in the other family - in 1 point (prove this!) Prove also that the zero locus of any two of the three quadratic polynomials defining C is the union of C and a line on Q meeting C in two points (or being tangent to C). How many such tangent lines to C are there in the family consisting generically of lines meeting C in two points?

An alternative description of C is the image of the *Veronese embedding* of $\mathbb{P}^1$ in $\mathbb{P}^3$:

$$C = \{[X^3, X^2Y, XY^2, Y^3] \mid [X, Y] \in \mathbb{P}^1\} = \nu_3(\mathbb{P}^1).$$

Show that this is indeed an embedding, and that its image indeed coincides with the twisted cubic curve C.

Example 3. What are the hyperplane sections of Q, i.e. $\mathbb{P}^2 \cap Q$, as the hyperplane $\mathbb{P}^2$ varies in $\mathbb{P}^3$?

The strategy here is to restrict the equation of Q to the hyperplane $\mathbb{P}^2$, and to realize that a homogeneous quadratic polynomial in 3 variables is either irreducible (smooth plane conic hyperplane section of Q), or factors as the product of two homogeneous linear factors (the hyperplane section here is the union of two intersecting lines in $\mathbb{P}^2$). Prove that we will never get the quadratic polynomial to factor as a perfect square of a linear form (i.e. no hyperplane in $\mathbb{P}^3$ intesects Q in a "double line").

We push the above considerations an inch further to ask the following question: can we construct a variety $\mathcal{H}_Q$ which in some reasonable way will be the family of *all* hyperplane sections of Q? In other words, can we separate all hyperplane sections of Q, so that they do not intersect anymore, but stay as fibers of some map? The answer is "YES", yet we have to work a bit to construct this variety.

For starters, we can construct the *universal hyperplane* $\mathcal{H}$ in $\mathbb{P}^3$ – apriori, we want this to be a variety, representing *all* hyperplanes in $\mathbb{P}^3$, in other words, $\mathcal{H}$ should be a family of all hyperplanes in $\mathbb{P}^3$. The first step is to realize what variety *parametrizes* these hyperplanes – this is the *dual* $(\mathbb{P}^3)^*$, which is really $\mathbb{P}^3$ all over again, by with different coordinates. If $\mathbb{P}^3$ has coordinates $[Z_0, Z_1, Z_2, Z_3]$, then a hyperplane in $\mathbb{P}^3$ is given by a linear form $W_0Z_0 + W_1Z_1 + W_2Z_2 + W_3Z_3 = 0$ for some fixed $[W_0, W_1, W_2, W_3] \in (\mathbb{P}^3)^*$. Thus, the dual $(\mathbb{P}^3)^*$ has coordinates W_0, W_1, W_2, W_3; points in $(\mathbb{P}^3)^*$ correspond to hyperplanes in $\mathbb{P}^3$, and hyperlanes in $(\mathbb{P}^3)^*$ correspond to points in $\mathbb{P}^3$.

Now, the universal hyperplane $\mathcal{H}$ should be defined as a set by

$$\mathcal{H} = \{[p, H] \mid p \in \mathbb{P}^3,\ H \in (\mathbb{P}^3)^*,\ p \in H\}.$$

Therefore, we naturally consider $\mathcal{H} \in \mathbb{P}^3 \times (\mathbb{P}^3)^*$ - it is the zero locus of a single *bihomogeneous* polynomial:

$$W_0Z_0 + W_1Z_1 + W_2Z_2 + W_3Z_3 = 0.$$

In terms of the coordinates T_{ij} on $\mathbb{P}^{15}$, in which $\mathbb{P}^3 \times (\mathbb{P}^3)^*$ is embedded, this polynomial reads:

$$T_{00} + T_{11} + T_{22} + T_{33} = 0.$$

We conclude that $\mathcal{H}$ is a subvariety of the product $\mathbb{P}^3 \times (\mathbb{P}^3)^*$; more precisely, it is a hyperplane section of $\mathcal{H}$ inside $\mathbb{P}^{15}$.

Finally, to construct the *universal hyperplane section* $\mathcal{H}_Q$ of the quadric Q, we only have to intersect with $\mathcal{H}$:

$$\mathcal{H}_Q = Q \cap \mathcal{H} \subset \mathbb{P}^3 \times (\mathbb{P}^3)^*.$$

Since the product on the right has two natural projections π_1 and π_2 onto the two factors $\mathbb{P}^3$ and $(\mathbb{P}^3)^*$, we can restrict these maps to $\mathcal{H}_Q$, and ask what the fibers of π_1 and π_2 are. Prove that the fibers of π_1 are all (isomorphic to) $\mathbb{P}^3$, while the fibers of π_2 are the hyperplane sections of Q – what we wanted in the first place.

Maps of varieties $\phi : X \to Y$ whose fibers are all isomorphic to some $\mathbb{P}^k$ are called $\mathbb{P}^k$– *bundles over* Y. Sometimes it is important to classify all such bundles over a fixed variety Y – this describes additional invariants of Y, which may be used for instance to identify two non–isomorphic varieties.

Example 4. As we saw above, all hyperplanes in $\mathbb{P}^3$ can be parametrized by the variety $(\mathbb{P}^3)^*$ (which is isomorphic to $\mathbb{P}^3$.) In this case, the points of $(\mathbb{P}^3)^*$ are in 1–1 correspondence with the hyperplanes in question, and $(\mathbb{P}^3)^*$ reflects (in a certain sense) how the hyperplanes vary in $\mathbb{P}^3$ - that is, for any "nice" family $\mathcal{F}$ of hyperplanes in $\mathbb{P}^3$ the subset of $(\mathbb{P}^3)^*$ corresponding to $\mathcal{F}$ is a *subvariety* of $(\mathbb{P}^3)^*$. (The word "nice" has a very technical meaning, usually called "flatness" of families. We shall not discuss this here since it will take us too far afield.)

One can easily generalize the above construction to parametrize all hyperplanes in $\mathbb{P}^n$ by the dual projective space $(\mathbb{P}^n)^*$. A natural question arises: Can we find varieties parametrizing other objects, say, conics in $\mathbb{P}^2$? Such varieties are called *parameter spaces.*

Let $\mathcal{P}$ be the set of all conics in $\mathbb{P}^2$. If we fix the coordinates of $\mathbb{P}^2$ to be X, Y, Z, then a conic C is determined upto a scalar by a quadratic equation:

$$a_0X^2 + a_1Y^2 + a_2Z^2 + a_3XY + a_4YZ + a_5ZX = 0.$$

Such an equation determines a projective point $[a_0, a_1, ..., a_5] \in \mathbb{P}^5$, and conversely, any point of $\mathbb{P}^5$ determines a unique quadratic polynomial (upto a scalar). Thus, the parameter space for all conics in $\mathbb{P}^2$ is $\mathcal{P} = \mathbb{P}^5$.

Similarly, the parameter space of all hypersurfaces of degree d in $\mathbb{P}^n$ (i.e. subvarieties given by single degree d homogeneous polynomials on $\mathbb{P}^n$) is $\mathcal{P}_{d,n} = \mathbb{P}^N$ where $N = \binom{n+d}{d} - 1$. Note a slight technicality here: we have included as points in $\mathcal{P}_{d,n}$ "hypersurfaces" corresponding to polynomials with multiple factors. For example, in the case of conics in $\mathbb{P}^2$, we included as points in $\mathcal{P}_{2,2}$ all "double" lines. One can show that the set of such "multiple" (or more precisely, *non-reduced*) hypersurfaces is in fact a subvariety of the corresponding parameter space $\mathcal{P}_{d,n}$.

Example 5. Parameter spaces parametrize usually not just objects X sharing some common properties, but also the embeddings of X in projective space. For example, there are really only three types of conics in $\mathbb{P}^2$ – the irreducible (smooth) conics, the joins of two different lines, and the double lines. Every irreducible conic can be transformed into any other irreducible conic after a suitable change of variables (coordinate change) on $\mathbb{P}^2$, etc. Thus, in constructing $\mathcal{P}_{2,2}$, we grossly "overcounted" the irreducible conics (well, we were parametrizing, therefore, not just the conics, but the pairs (C, ϕ) where C is a plane conic and $\phi : C \hookrightarrow \mathbb{P}^2$ is an embedding.)

The philosophy of viewing a variety as an object with a given embedding in some $\mathbb{P}^n$ is inherent to XIX century algebraic geometry, especially to the Italian school. XX century changed this view by considering varieties as objects on their own, disregarding particular embeddings in projective space. For example, any irreducible conic in $\mathbb{P}^2$ is really a $\mathbb{P}^1$ embedded in a certain way in $\mathbb{P}^2$:

$$\nu_2 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2, \ \nu_2([X, Y]) = [X^2, XY, Y^2].$$

Similarly, the twisted cubic C in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1$: $C = \nu_3(\mathbb{P}^3)$. We say that these curves are isomorphic to each other because there exist nicely defined maps via polynomials going back and forth between these varieties, whose compositions are identities. Thus, the *intrinsic* properties of $\mathbb{P}^1$ are preserved under these isomorphism, and therefore the embeddings do not change the actual variety.

Some *extrinsic* properties, however, change, and these cause the different embeddings of $\mathbb{P}^1$ to look different. For example, define the *degree* of $X \cong \mathbb{P}^1 \subset \mathbb{P}^n$ to be the number of points in the intersection of a general hyperplane in $\mathbb{P}^n$ with X. Thus, the conics in $\mathbb{P}^2$ have degree 2, and will keep their degree if we embed now $\mathbb{P}^2$ as a linear subspace of a bigger $\mathbb{P}^n$. However, the twisted cubic C in $\mathbb{P}^3$ has degree 3 (one way to see this is to recall that a line in one ruling meets C in 1 point, while a line in the other ruling meets C in 2 points.)

While parameter spaces may take into account such extrinsic properties as *degrees* of varieties, *moduli spaces* usually parametrize objects according to only their intrinsic properties, and hence are much harder to be constructed. To even state what common instrinsic properties can be characterized will take too much ink on this handout. But let us mention one very famous example – the moduli space $\mathfrak{M}_g$ of *smooth curves of genus* g. [1] These curves do not lie (and cannot be embedded in general) in the same projective space $\mathbb{P}^n$. The best we can say is that each such (non–hyperelliptic) curve can be embedded in $\mathbb{P}^{2g-3}$, but we don't care about these embeddings anyways. Yet, $\mathfrak{M}_g$ can be constructed, and it is a variety of dimension $3g-3$ for $g \geq 2$. For $g=1$ there is only one such curve – $\mathbb{P}^1$, so $\mathfrak{M}_1$ is really just one point; for $g=1$ – the elliptic curves can be effectively parametrized by a certain cross ratio, and hence $\mathfrak{M}_1 \cong \mathbb{P}^1$.

A further development of this theory is the *Deligne-Mumford* compactification of $\mathfrak{M}_\mathfrak{g}$. Since $\mathfrak{M}_g$ is *not* a projective variety, one can have a nice family of *smooth curves* degenerating to a singular curve, but $\mathfrak{M}_g$ does not have any points to reflect the limiting singular member of the family. The question arises – what is the "minimal" set of *singular curves* must be added to the set of smooth curves in order to obtain a "nice" moduli space $\overline{\mathfrak{M}}_g$, compactifying $\mathfrak{M}_g$? Deligne and Mumford chose (and for very good reasons) the set of the so–called *stable* curves C of genus g. These are connected curves with at most *nodal* type of singularities (e.g. take two lines intersecting in $\mathbb{P}^2$), and such that if they contain a $\mathbb{P}^1$–component, then the latter must meet at least 3 other components of the curve. The last condition is added to ensure that the curves have finite groups of automorphisms. With this said, $\overline{\mathfrak{M}}_g$ is *the moduli space of all stable curves of genus* g. It is a projective variety which contains $\mathfrak{M}_g$ as an open dense set, and it reflects naturally the variation of "nice" families of stable curves. Moreover, any "nice" family whose general members are smooth curves, but whose special members can be as nasty as you wish, can be brought in

[1] Be forwarned that what follows in this section is littered with too many unexplained terms in order for the text to be self–contained.

an essentially one way to a family with only stable members. This process is called *semistable reduction* and it is the basis for many related constructions in algebraic geometry.

Zariski Topology

Definitions.

(a) An ideal P of a ring R is called *prime* if whenever $ab \in P$ for $a, b \in R$, then $a \in P$ or $b \in P$ (or both).

(b) A variety X is called *irreducible* if for any decomposition $X = X_1 \cup X_2$ of X into a union of two subvarieties, either $X_1 = X$ or $X_2 = X$. In other words, there are no non–trivial decompositions of X into smaller varieties.

Proposition 2. *A variety X is irreducible iff its ideal $I(X)$ is prime.*

Thus, there exists a one–to–one correspondence between the set of varieties $\mathcal{X}$ and the set of prime (homogeneous if projective X) ideals $\mathcal{P}$ in the corresponding polynomial ring.

Theorem 4. *Any radical ideal $I \subset K[x_1, ..., x_n]$ is uniquely expressible as a finite intersection of prime ideals P_i with $P_i \not\subset P_j$ for $i \neq j$. Equivalently, any variety X can be uniquely expressed as a finite union of irreducible subvarieties X_i with $X_i \not\subset X_j$ for $i \neq j$.*

The varieties X_i appearing in this unique decomposition are called the *irreducible components* of X.

Definition. Let X be a set of points in some space. A *topology* on X is a set $\mathcal{T}$ of designated subsets of X, called the *open* sets of X, so that the following axioms are satisfied:

(a) The union of any collection of open sets is open.

(b) The intersection of any finite collection of open sets is open.

(c) X and $\emptyset$ are open.

The *closed* sets in X are the complements of the open sets.

We define below the so–called *Zariski topology* on algebraic varieties. If we work over $\mathbb{C}$, every variety can be roughly viewed as a *complex manifold* X (with the exception of a proper subset of *its singular points*). Through its embedding in, say, $\mathbb{C}^n$, X will inherit the usual *complex analytic* topology from $\mathbb{C}^n$ – a basis for the open sets on X will consist of the intersections of X with any finite balls in $\mathbb{C}^n$.

The *Zariski topology* is a different kind of topology. A basis for the open sets in X is given by the sets

$$U_f = \{p \in X \mid f(p) \neq 0\}$$

where f ranges over polynomials (homogeneous if projective X).

Lemma 2. *The Zariski topology is indeed a topology on X.*

Exercise. Show that the Zariski topology on the projective line $\mathbb{P}^1_{\mathbb{C}}$ is different from the analytic topology of $\mathbb{P}^1_{\mathbb{C}}$.

Many statements in algebraic geometry are true for *general* points on varieties, i.e. if X is a variety and U is an open dense set of X, then any point $p \in U$ is called a *general* point on U. (If X is irreducible, then any nonempty open set will be dense. This, in particular, makes Zariski topology a *non-Housdorff* topology – in the latter, one needs for any two points of X to have two nonintersecting open sets containing each one of the points. This confirms once again that the Zariski topology is much coarser than the analytic topology.)

2. Bezout's Theorem

Definition. Suppose that X and $Y \subset \mathbb{P}^n$ are two irreducible varieties and that their intersection has irreducible components Z_i. We say that X and Y intersect *generically transversally* if, for each i, X and Y intersect transversally at a general point $p_i \in Z_i$, i.e., are smooth at p_i with tangent spaces spanning $\mathbb{T}_{p_i}(\mathbb{P}^n)$ (the tangent space to $\mathbb{P}^n$ at p_i.)

Theorem 5. *(Bezout) Let X and $Y \subset \mathbb{P}^n$ be subvarieties of pure dimensions k and l with $k + l \geq n$, and suppose they intersect generically transversely. Then*

$$deg\,(X \cap Y) = deg(X) \cdot deg(Y).$$

In particular, if $k + l = n$, this says that $X \cap Y$ will consist of $deg(X) \cdot deg(Y)$ points.

A pair of pure–dimensional varieties X and $Y \subset \mathbb{P}^n$ *intersect properly* if their intersection has the expected dimension, i.e.,

$$\dim(X \cap Y) = \dim(X) + \dim(Y) - n.$$

Theorem 6. *If X and Y intersect properly,*

$$deg(X) \cdot deg(Y) = \sum m_Z(X, Y) \cdot deg(Z)$$

where the sum is over all irreducible subvarieties Z of the appropriate dimension (in effect, over all irreducible components Z of $X \cap Y$). Here $m_Z(X, Y)$ is the intersection multiplicity of X and Y along Z:

1. *$m_Z(X, Y) \geq 1$ for all $Z \subset X \cap Y$ ($m_Z(X, Y) = 0$ otherwise.)*
2. *$m_Z(X, Y) = 1$ if X and Y intersect transversely at a general point of Z.*
3. *$m_Z(X, Y)$ is additive, i.e. $m_Z(X \cup X', Y) = m_Z(X, Y) + m_Z(X', Y)$ for any X and X' as long as all three numbers are defined and X and X' have no common components.*

In particular, for any subvarieties X and Y of pure dimension in $\mathbb{P}^n$ intersecting properly:

$$\deg(X \cap Y) \leq \deg(X) \cdot \deg(Y).$$