

# LINEAR RECURSIVE SEQUENCES

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## 1. SEQUENCES

A *sequence* is an infinite list of numbers, like

$$(1) \quad 1, 2, 4, 8, 16, 32, \dots$$

The numbers in the sequence are called its *terms*. The general form of a sequence is

$$a_1, a_2, a_3, \dots$$

where  $a_n$  is the  $n$ -th term of the sequence. In the example (1) above,  $a_1 = 1$ ,  $a_2 = 2$ ,  $a_3 = 4$ , and so on.

The notations  $\{a_n\}$  or  $\{a_n\}_{n=1}^{\infty}$  are abbreviations for

$$a_1, a_2, a_3, \dots$$

Occasionally the indexing of the terms will start with something other than 1. For example,  $\{a_n\}_{n=0}^{\infty}$  would mean

$$a_0, a_1, a_2, \dots$$

(In this case  $a_n$  would be the  $(n + 1)$ -st term.)

For some sequences, it is possible to give an *explicit formula* for  $a_n$ : this means that  $a_n$  is expressed as a function of  $n$ . For instance, the sequence (1) above can be described by the explicit formula  $a_n = 2^{n-1}$ .

## 2. RECURSIVE DEFINITIONS

An alternative way to describe a sequence is to list a few terms and to give a rule for computing the rest of the sequence. Our example (1) above can be described by the starting value  $a_1 = 1$  and the rule  $a_{n+1} = 2a_n$  for integers  $n \geq 1$ . Starting from  $a_1 = 1$ , the rule implies that

$$a_2 = 2a_1 = 2(1) = 2$$

$$a_3 = 2a_2 = 2(2) = 4$$

$$a_4 = 2a_3 = 2(4) = 8,$$

and so on; each term in the sequence can be computed recursively in terms of the terms previously computed. A rule such as this giving the next term in terms of earlier terms is also called a *recurrence relation* (or simply *recurrence*).

### 3. LINEAR RECURSIVE SEQUENCES

A sequence  $\{a_n\}$  is said to satisfy the *linear recurrence* with coefficients  $c_k, c_{k-1}, \dots, c_0$  if

$$(2) \quad c_k a_{n+k} + c_{k-1} a_{n+k-1} + \dots + c_1 a_{n+1} + c_0 a_n = 0$$

holds for all integers  $n$  for which this makes sense. (If the sequence starts with  $a_1$ , then this means for  $n \geq 1$ .) The integer  $k$  is called the *order* of the linear recurrence.

A *linear recursive sequence* is a sequence of numbers  $a_1, a_2, a_3, \dots$  satisfying some linear recurrence as above with  $c_k \neq 0$  and  $c_0 \neq 0$ . For example, the sequence (1) satisfies

$$a_{n+1} - 2a_n = 0$$

for all integers  $n \geq 1$ , so it is a linear recursive sequence satisfying a recurrence of order 1, with  $c_1 = 1$  and  $c_0 = -2$ .

Requiring  $c_k \neq 0$  guarantees that the linear recurrence can be used to express  $a_{n+k}$  as a linear combination of earlier terms:

$$a_{n+k} = -\frac{c_{k-1}}{c_k} a_{n+k-1} - \dots - \frac{c_1}{c_k} a_{n+1} - \frac{c_0}{c_k} a_n.$$

The requirement  $c_0 \neq 0$  lets one express  $a_n$  as a linear combination of *later* terms:

$$a_n = -\frac{c_k}{c_0} a_{n+k} - \frac{c_{k-1}}{c_0} a_{n+k-1} - \dots - \frac{c_1}{c_0} a_{n+1}.$$

This lets one *define*  $a_0, a_{-1}$ , and so on, to obtain a *doubly infinite sequence*

$$\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots$$

that now satisfies the same linear recurrence for *all* integers  $n$ , positive or negative.

### 4. CHARACTERISTIC POLYNOMIALS

The *characteristic polynomial* of a linear recurrence

$$c_k a_{n+k} + c_{k-1} a_{n+k-1} + \dots + c_1 a_{n+1} + c_0 a_n = 0$$

is defined to be the polynomial

$$c_k x^k + c_{k-1} x^{k-1} + \dots + c_1 x + c_0.$$

For example, the characteristic polynomial of the recurrence  $a_{n+1} - 2a_n = 0$  satisfied by the sequence (1) is  $x - 2$ .

Here is another example: the famous *Fibonacci sequence*

$$\{F_n\}_{n=0}^\infty = 0, 1, 1, 2, 3, 5, 8, 13, \dots$$

which can be described by the starting values  $F_0 = 0, F_1 = 1$  and the recurrence relation

$$(3) \quad F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 2.$$

To find the characteristic polynomial, we first need to rewrite the recurrence relation in the form (2). The relation (3) is equivalent to

$$(4) \quad F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.$$

Rewriting it as

$$(5) \quad F_{n+2} - F_{n+1} - F_n = 0$$

shows that  $\{F_n\}$  is a linear recursive sequence satisfying a recurrence of order 2, with  $c_2 = 1, c_1 = -1$ , and  $c_0 = -1$ . The characteristic polynomial is  $x^2 - x - 1$ .

5. IDEALS AND MINIMAL CHARACTERISTIC POLYNOMIALS

The same sequence can satisfy many different linear recurrences. For example, doubling (5) shows the Fibonacci sequence also satisfies

$$2F_{n+2} - 2F_{n+1} - 2F_n = 0,$$

which is a linear recurrence with characteristic polynomial  $2x^2 - 2x - 2$ . It also satisfies

$$F_{n+3} - F_{n+2} - F_{n+1} = 0,$$

and adding these two relations, we find that  $\{F_n\}$  also satisfies

$$F_{n+3} + F_{n+2} - 3F_{n+1} - 2F_n = 0$$

which has characteristic polynomial  $x^3 + x^2 - 3x - 2 = (x + 2)(x^2 - x - 1)$ .

Now consider an arbitrary sequence  $\{a_n\}$ . Let  $I$  be the set of characteristic polynomials of *all* linear recurrences satisfied by  $\{a_n\}$ . Then

- (a) If  $f(x) \in I$  and  $g(x) \in I$  then  $f(x) + g(x) \in I$ .
- (b) If  $f(x) \in I$  and  $h(x)$  is any polynomial, then  $h(x)f(x) \in I$ .

In general, a nonempty set  $I$  of polynomials satisfying (a) and (b) is called an *ideal*.

**Fact from algebra:** Let  $I$  be an ideal of polynomials. Then either  $I = \{0\}$  or else there is a unique monic polynomial  $f(x) \in I$  such that

$$I = \text{the set of polynomial multiples of } f(x) = \{h(x)f(x) \mid h(x) \text{ is a polynomial}\}.$$

(A polynomial is *monic* if the coefficient of the highest power of  $x$  is 1.)

This fact, applied to the ideal of characteristic polynomials of a linear recursive sequence  $\{a_n\}$  shows that there is always a *minimal characteristic polynomial*  $f(x)$ , which is the monic polynomial of lowest degree in  $I$ . It is the characteristic polynomial of the lowest order nontrivial linear recurrence satisfied by  $\{a_n\}$ . The characteristic polynomial of any other linear recurrence satisfied by  $\{a_n\}$  is a polynomial multiple of  $f(x)$ .

The *order* of a linear recursive sequence  $\{a_n\}$  is defined to be the lowest order among all (nontrivial) linear recurrences satisfied by  $\{a_n\}$ . The order also equals the degree of the minimal characteristic polynomial. For example, as we showed above,  $\{F_n\}$  satisfies

$$F_{n+3} + F_{n+2} - 3F_{n+1} - 2F_n = 0,$$

but we also know that

$$F_{n+2} - F_{n+1} - F_n = 0,$$

and it is easy to show that  $\{F_n\}$  cannot satisfy a linear recurrence of order less than 2, so  $\{F_n\}$  is a linear recursive sequence of order 2, with minimal characteristic polynomial  $x^2 - x - 1$ .

6. THE MAIN THEOREM

**Theorem 1.** Let  $f(x) = c_k x^k + \dots + c_0$  be a polynomial with  $c_k \neq 0$  and  $c_0 \neq 0$ . Factor  $f(x)$  over the complex numbers as

$$f(x) = c_k(x - r_1)^{m_1}(x - r_2)^{m_2} \dots (x - r_\ell)^{m_\ell},$$

where  $r_1, r_2, \dots, r_\ell$  are distinct nonzero complex numbers, and  $m_1, m_2, \dots, m_\ell$  are positive integers. Then a sequence  $\{a_n\}$  satisfies the linear recurrence with characteristic polynomial  $f(x)$  if and only if there exist polynomials  $g_1(n), g_2(n), \dots, g_\ell(n)$  with  $\deg g_i \leq m_i - 1$  for  $i = 1, 2, \dots, \ell$  such that

$$a_n = g_1(n)r_1^n + \dots + g_\ell(n)r_\ell^n \quad \text{for all } n.$$

Here is an important special case.

**Corollary 2.** *Suppose in addition that  $f(x)$  has no repeated factors; in other words suppose that  $m_1 = m_2 = \dots = m_\ell = 1$ . Then  $f(x) = c_k(x - r_1)(x - r_2) \dots (x - r_\ell)$  where  $r_1, r_2, \dots, r_\ell$  are distinct nonzero complex numbers (the roots of  $f$ ). Then  $\{a_n\}$  satisfies the linear recurrence with characteristic polynomial  $f(x)$  if and only if there exist constants  $B_1, B_2, \dots, B_\ell$  (not depending on  $n$ ) such that*

$$a_n = B_1 r_1^n + \dots + B_\ell r_\ell^n \quad \text{for all } n.$$

## 7. EXAMPLE: SOLVING A LINEAR RECURRENCE

Suppose we want to find an explicit formula for the sequence  $\{a_n\}$  satisfying  $a_0 = 1$ ,  $a_1 = 4$ , and

$$(6) \quad a_{n+2} = \frac{a_{n+1} + a_n}{2} \text{ for } n \geq 0.$$

Since  $\{a_n\}$  satisfies a linear recurrence with characteristic polynomial  $x^2 - \frac{1}{2}x - \frac{1}{2} = (x - 1)(x + \frac{1}{2})$ , we know that there exist constants  $A$  and  $B$  such that

$$(7) \quad a_n = A(1)^n + B \left(-\frac{1}{2}\right)^n$$

for all  $n$ . The formula (7) is called the *general solution* to the linear recurrence (6). To find the *particular solution* with the correct values of  $A$  and  $B$ , we use the known values of  $a_0$  and  $a_1$ :

$$\begin{aligned} 1 = a_0 &= A(1)^0 + B \left(-\frac{1}{2}\right)^0 = A + B \\ 4 = a_1 &= A(1)^1 + B \left(-\frac{1}{2}\right)^1 = A - B/2. \end{aligned}$$

Solving this system of equations yields  $A = 3$  and  $B = -2$ . Thus the particular solution is

$$a_n = 3 - 2 \left(-\frac{1}{2}\right)^n.$$

(As a check, one can try plugging in  $n = 0$  or  $n = 1$ .)

## 8. EXAMPLE: THE FORMULA FOR THE FIBONACCI SEQUENCE

As we worked out earlier,  $\{F_n\}$  satisfies a linear recurrence with characteristic polynomial  $x^2 - x - 1$ . By the quadratic formula, this factors as  $(x - \alpha)(x - \beta)$  where  $\alpha = (1 + \sqrt{5})/2$  is the golden ratio, and  $\beta = (1 - \sqrt{5})/2$ . The main theorem implies that there are constants  $A$  and  $B$  such that

$$F_n = A\alpha^n + B\beta^n$$

for all  $n$ . Using  $F_0 = 0$  and  $F_1 = 1$  we obtain

$$0 = A + B, \quad 1 = A\alpha + B\beta.$$

Solving for  $A$  and  $B$  yields  $A = 1/(\alpha - \beta)$  and  $B = -1/(\alpha - \beta)$ , so

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

for all  $n$ .

### 9. EXAMPLE: FINDING A LINEAR RECURRENCE FROM AN EXPLICIT FORMULA

Let  $a_n = (n + 2^n)F_n$ , where  $\{F_n\}$  is the Fibonacci sequence. Then by the explicit formula for  $F_n$ ,

$$\begin{aligned} a_n &= (n + 2^n) \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \\ &= \left[ \left( \frac{1}{\alpha - \beta} \right) n \right] \alpha^n + \left[ \left( \frac{-1}{\alpha - \beta} \right) n \right] \beta^n + \left( \frac{1}{\alpha - \beta} \right) (2\alpha)^n + \left( \frac{-1}{\alpha - \beta} \right) (2\beta)^n. \end{aligned}$$

By Theorem 1,  $\{a_n\}$  satisfies a linear recurrence with characteristic polynomial

$$\begin{aligned} (x - \alpha)^2(x - \beta)^2(x - 2\alpha)(x - 2\beta) &= (x^2 - x - 1)^2 [x^2 - 2(\alpha + \beta)x + 4\alpha\beta] \\ &= (x^2 - x - 1)^2(x^2 - 2x - 4) \\ &= x^6 - 4x^5 - x^4 + 12x^3 + x^2 - 10x + 4, \end{aligned}$$

where we have used the identity  $x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - x - 1$  to compute  $\alpha + \beta$  and  $\alpha\beta$ . In other words,

$$a_{n+6} - 4a_{n+5} - a_{n+4} + 12a_{n+3} + a_{n+2} - 10a_{n+1} + 4a_n = 0$$

for all  $n$ . In fact, we have found the minimal characteristic polynomial, since if the actual minimal characteristic polynomial were a proper divisor of  $(x^2 - x - 1)^2(x^2 - 2x - 4)$ , then according to Theorem 1, the explicit formula for  $a_n$  would have had a different, simpler form.

### 10. INHOMOGENEOUS RECURRENCE RELATIONS

Suppose we wanted an explicit formula for a sequence  $\{a_n\}$  satisfying  $a_0 = 0$ , and

$$(8) \quad a_{n+1} - 2a_n = F_n \quad \text{for } n \geq 0,$$

where  $\{F_n\}$  is the Fibonacci sequence as usual. This is not a linear recurrence in the sense we have been talking about (because of the  $F_n$  on the right hand side instead of 0), so our usual method does not work. A recurrence of this type, linear except for a function of  $n$  on the right hand side, is called an *inhomogeneous recurrence*.

We can solve inhomogeneous recurrences explicitly when the right hand side is itself a linear recursive sequence. In our example,  $\{a_n\}$  also satisfies

$$(9) \quad a_{n+2} - 2a_{n+1} = F_{n+1}$$

and

$$(10) \quad a_{n+3} - 2a_{n+2} = F_{n+2}.$$

Subtracting (8) and (9) from (10) yields

$$a_{n+3} - 3a_{n+2} + a_{n+1} + 2a_n = F_{n+2} - F_{n+1} - F_n = 0.$$

Thus  $\{a_n\}$  is a linear recursive sequence after all! The characteristic polynomial of this new linear recurrence is  $x^3 - 3x^2 + x + 2 = (x - 2)(x^2 - x - 1)$ , so by Theorem 1, there exist constants  $A, B, C$  such that

$$a_n = A \cdot 2^n + B\alpha^n + C\beta^n$$

for all  $n$ . Now we can use  $a_0 = 0$ , and the values  $a_1 = 0$  and  $a_2 = 1$  obtained from (8) to determine  $A, B, C$ . After some work, one finds  $A = 1$ ,  $B = -\alpha^2/(\alpha - \beta)$ , and  $C = \beta^2/(\alpha - \beta)$ , so  $a_n = 2^n - F_{n+2}$ .

If  $\{x_n\}$  is any other sequence satisfying

$$(11) \quad x_{n+1} - 2x_n = F_n$$

but not necessarily  $x_0 = 0$ , then subtracting (8) from (11) shows that the sequence  $\{y_n\}$  defined by  $y_n = x_n - a_n$  satisfies  $y_{n+1} - 2y_n = 0$  for all  $n$ , so  $y_n = D \cdot 2^n$  for some number  $D$ . Hence the *general solution* of (11) has the form

$$x_n = 2^n - F_{n+2} + D \cdot 2^n,$$

or more simply,

$$x_n = E \cdot 2^n - F_{n+2},$$

where  $E$  is some constant.

In general, this sort of argument proves the following.

**Theorem 3.** *Let  $\{b_n\}$  be a linear recursive sequence satisfying a recurrence with characteristic polynomial  $f(x)$ . Let  $g(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0$  be a polynomial. Then every solution  $\{x_n\}$  to the inhomogeneous recurrence*

$$(12) \quad c_k x_{n+k} + c_{k-1} x_{n+k-1} + \cdots + c_1 x_{n+1} + c_0 x_n = b_n$$

*also satisfies a linear recurrence with characteristic polynomial  $f(x)g(x)$ .*

*Moreover, if  $\{x_n\} = \{a_n\}$  is one particular solution to (12), then all solutions have the form  $x_n = a_n + y_n$ , where  $\{y_n\}$  ranges over the solutions of the linear recurrence*

$$c_k y_{n+k} + c_{k-1} y_{n+k-1} + \cdots + c_1 y_{n+1} + c_0 y_n = 0.$$

## 11. THE MAHLER-LECH THEOREM

Here is a deep theorem about linear recursive sequences:

**Theorem 4** (Mahler-Lech theorem). *Let  $\{a_n\}$  be a linear recursive sequence of complex numbers, and let  $c$  be a complex number. Then there exists a finite (possibly empty) list of arithmetic progressions  $T_1, T_2, \dots, T_m$  and a finite (possibly empty) set  $S$  of integers such that*

$$\{n \mid a_n = c\} = S \cup T_1 \cup T_2 \cup \cdots \cup T_m.$$

Warning: don't try to prove this at home! This is *very* hard to prove. The proof uses “ $p$ -adic numbers.”

12. PROBLEMS

There are a lot of problems here. Just do the ones that interest you.

- (1) If the Fibonacci sequence is extended to a doubly infinite sequence satisfying the same linear recurrence, then what will  $F_{-4}$  be? (Is it easier to do this using the recurrence, or using the explicit formula?)
- (2) Find the smallest degree polynomial that could be the minimal characteristic polynomial of a sequence that begins

$$2, 5, 18, 67, 250, 933, \dots$$

Assuming that the sequence *is* a linear recursive sequence with this characteristic polynomial, find an explicit formula for the  $n$ -th term.

- (3) Suppose that  $a_n = n^2 + 3n + 7$  for  $n \geq 1$ . Prove that  $\{a_n\}$  is a linear recursive sequence, and find its minimal characteristic polynomial.
- (4) Suppose  $a_1 = a_2 = a_3 = 1$ ,  $a_4 = 3$ , and  $a_{n+4} = 3a_{n+2} - 2a_n$  for  $n \geq 1$ . Prove that  $a_n = 1$  if and only if  $n$  is odd or  $n = 2$ . (This is an instance of the Mahler-Lech theorem: for this sequence, one would take  $S = \{2\}$  and  $T_1 = \{1, 3, 5, 7, \dots\}$ .)
- (5) Suppose  $a_0 = 2$ ,  $a_1 = 5$ , and  $a_{n+2} = (a_{n+1})^2(a_n)^3$  for  $n \geq 0$ . (This is a recurrence relation, but not a linear recurrence relation.) Find an explicit formula for  $a_n$ .
- (6) Suppose  $\{a_n\}$  is a sequence such that  $a_{n+2} = a_{n+1} - a_n$  for all  $n \geq 1$ . Given that  $a_{38} = 7$  and  $a_{55} = 3$ , find  $a_1$ . (Hint: it is possible to solve this problem with very little calculation.)
- (7) Let  $\theta$  be a fixed real number, and let  $a_n = \cos(n\theta)$  for integers  $n \geq 1$ . Prove that  $\{a_n\}$  is a linear recursive sequence, and find the minimal characteristic polynomial. (Hint: if you know the definition of  $\cos x$  in terms of complex exponentials, use that. Otherwise, use the sum-to-product rule for the sum of cosines  $\cos(n\theta) + \cos((n+2)\theta)$ . For most but not all  $\theta$ , the degree of the minimal characteristic polynomial will be 2.)
- (8) Give an example of a sequence that is *not* a linear recursive sequence, and prove that it is not one.
- (9) Given a finite set  $S$  of positive integers, show that there exists a linear recursive sequence

$$a_1, a_2, a_3, \dots$$

such that  $\{n \mid a_n = 0\} = S$ .

- (10) A student tosses a fair coin and scores one point for each head that turns up, and two points for each tail. Prove that the probability of the student scoring  $n$  points at some time in a sequence of  $n$  tosses is  $\frac{1}{3} \left(2 + \left(-\frac{1}{2}\right)^n\right)$ .
- (11) Let  $F_n$  denote the  $n$ -th Fibonacci number. Let  $a_n = (F_n)^2$ . Prove that  $a_1, a_2, a_3, \dots$  is a linear recursive sequence, and find its minimal characteristic polynomial.
- (12) Prove the “fact from algebra” mentioned above in Section 5. (Hint: if  $I \neq \{0\}$ , pick a nonzero polynomial in  $I$  of smallest degree, and multiply it by a constant to get a monic polynomial  $f(x)$ . Use long division of polynomials to show that anything else in  $I$  is a polynomial multiple of  $f(x)$ .)
- (13) Suppose that  $a_1, a_2, \dots$  is a linear recursive sequence. For  $n \geq 1$ , let  $s_n = a_1 + a_2 + \dots + a_n$ . Prove that  $\{s_n\}$  is a linear recursive sequence.
- (14) Suppose  $\{a_n\}$  and  $\{b_n\}$  are linear recursive sequences. Let  $c_n = a_n + b_n$  and  $d_n = a_n b_n$  for  $n \geq 1$ .

- (a) Prove that  $\{c_n\}$  and  $\{d_n\}$  also are linear recursive sequences.  
 (b) Suppose that the minimal characteristic polynomials for  $\{a_n\}$  and  $\{b_n\}$  are  $x^2 - x - 2$  and  $x^2 - 5x + 6$ , respectively. What are the possibilities for the minimal characteristic polynomials of  $\{c_n\}$  and  $\{d_n\}$ ?  
 (15) Suppose that  $\{a_n\}$  and  $\{b_n\}$  are linear recursive sequences. Prove that

$$a_1, b_1, a_2, b_2, a_3, b_3, \dots$$

also is a linear recursive sequence.

- (16) Use the Mahler-Lech theorem to prove the following generalization.

Let  $\{a_n\}$  be a linear recursive sequence of complex numbers, and let  $p(x)$  be a polynomial. Then there exists a finite (possibly empty) list of arithmetic progressions  $T_1, T_2, \dots, T_m$  and a finite (possibly empty) set  $S$  of integers such that

$$\{n \mid a_n = p(n)\} = S \cup T_1 \cup T_2 \cup \dots \cup T_m.$$

(Hint: let  $b_n = a_n - p(n)$ .)

- (17) (1973 USAMO, no. 2) Let  $\{X_n\}$  and  $\{Y_n\}$  denote two sequences of integers defined as follows:

$$X_0 = 1, X_1 = 1, X_{n+1} = X_n + 2X_{n-1} \quad (n = 1, 2, 3, \dots),$$

$$Y_0 = 1, Y_1 = 7, Y_{n+1} = 2Y_n + 3Y_{n-1} \quad (n = 1, 2, 3, \dots).$$

Thus, the first few terms of the sequences are:

$$X : 1, 1, 3, 5, 11, 21, \dots,$$

$$Y : 1, 7, 17, 55, 161, 487, \dots$$

Prove that, except for the “1,” there is no term which occurs in both sequences.

- (18) (1963 IMO, no. 4) Find all solutions  $x_1, x_2, x_3, x_4, x_5$  to the system

$$x_5 + x_2 = yx_1$$

$$x_1 + x_3 = yx_2$$

$$x_2 + x_4 = yx_3$$

$$x_3 + x_5 = yx_4$$

$$x_4 + x_1 = yx_5,$$

where  $y$  is a parameter. (Hint: define  $x_6 = x_1, x_7 = x_2$ , etc., and find two different linear recurrences satisfied by  $\{x_n\}$ .)

- (19) (1967 IMO, no. 6) In a sports contest, there were  $m$  medals awarded on  $n$  successive days ( $n > 1$ ). On the first day, one medal and  $1/7$  of the remaining  $m - 1$  medals were awarded. On the second day, two medals and  $1/7$  of the now remaining medals were awarded; and so on. On the  $n$ -th and last day, the remaining  $n$  medals were awarded. How many days did the contest last, and how many medals were awarded altogether?

item (1974 IMO, no. 3) Prove that the number  $\sum_{k=0}^n \binom{2n+1}{k+1} 2^{3k}$  is not divisible by 5 for any integer  $n \geq 0$ .

- (20) (1980 USAMO, no. 3) Let  $F_r = x^r \sin(rA) + y^r \sin(rB) + z^r \sin(rC)$ , where  $x, y, z, A, B, C$  are real and  $A + B + C$  is an integral multiple of  $\pi$ . Prove that if  $F_1 = F_2 = 0$ , then  $F_r = 0$  for all positive integral  $r$ .



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