Berkeley Math Circle: Monthly Contest 7 Solutions

1. The numbers 1 through 7 are written on a blackboard. Each minute, two numbers are erased and their sum is written instead. Find all possible values for the final number left on the board.

   Solution. This number must be 28 no matter what order the numbers are erased. To see this, note that the sum of all the numbers on the board can never change. Since the initial sum is
   $$1 + 2 + \cdots + 7 = 28,$$
   this must also be the final sum and thus the final number left on the board. □

2. Show that for any positive integer n, the numbers $3n+2$ and $4n+3$ have no common factors greater than 1.

   Solution. Note that any common factor of the two numbers must also divide
   $$3(4n + 3) - 4(3n + 2) = (12 + 9) - (12 + 8) = 1,$$
   thus the only common divisor is 1. □

3. A *partition* of a positive integer n is a way of writing n as an unordered sum of not necessarily distinct positive integer parts. Show that the number of partitions of n with all odd parts equals the number of partitions with all distinct parts.

   Solution. We construct a bijection as follows: given a partition with all distinct parts, divide each even part into two equal parts, and repeat this until only odd parts are remaining. We can reverse this process by starting with a partition into all odd parts and repeatedly combining any two equal parts until all remaining parts are distinct. These operations are inverses, so this is indeed a bijection, and so the two sets of partitions have the same size. □

4. We have 2009 prime numbers $p_1 < p_2 < p_3 < \cdots < p_{2009}$ such that $p_1^2+p_2^2+\cdots+p_{2009}^2$ is a perfect square. Prove that p_1 divides $p_{2009}^2 - p_{2008}^2$.

   Solution. Note that $p_i^2 \equiv 1 \pmod 3$ whenever $p_i \neq 3$. We thus claim that 3 is one of the prime numbers. If not, the sum is $2009 \equiv 2 \pmod 3$, contradiction.

   If $p_1 = 2$ there is nothing to prove. Otherwise, if $p_1 = 3$, then the last comment finishes. □

5. Two numbers are *relatively prime* if their only common divisor is 1. For a positive integer n, let $\phi(n)$ be the number of positive integers less than or equal to n and relatively prime to n. Write $d \mid n$ if d is a divisor of n. Find
   $$\sum_{d \mid n} \phi(d).$$

In other words, compute the sum of $\phi(d)$ across all divisors of n.

Solution. We claim that $\sum_{d|n} \phi(n) = n$. To see this, consider the fractions

$$\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n},$$

and write them all in simplest form. There are n fractions total, and for each $d|n$, exactly d of them will have denominator d. The desired sum follows. □

6. Define the function $s\colon \mathbb{Z}^2 \to \mathbb{Z}$ by

$$s(n,k) = \begin{cases} 1 & n \le k \\ -1 & n > k. \end{cases}$$

Prove that if integers $x_1, \dots, x_{100}$ satisfy $x_i^2 = 1$ for each i, then

$$\prod_{n=1}^{100} \left(\sum_{k=1}^{100} s(n,k) x_k \right) = 0.$$

Solution. The is a consequence of the so-called Discrete Intermediate Value Theorem. Define

$$S_n = \frac{1}{2} \sum_{k=1}^{100} s(n,k) x_k$$

for $n = 0, 1, 2, \dots, 100$, and each is an integer. Observe that

$$|S_{n+1} - S_n| = 1.$$

for any $n = 0, 1, 2, \dots, 99$. On the other hand $S_{100} = -S_0$. If $S_0 = S_{100} = 0$ then the conclusion is clear. Otherwise, there must be some intermediate index k with $S_k = 0$, finishing the problem. □

7. Let Ω be a fixed circle and $\overline{BC}$ a fixed chord of that circle which is not a diameter. A variable diameter $\overline{AD}$ of Ω, with A on minor arc $\widehat{BC}$, is chosen. Line BD meets line AC at E, while line CD meets line AB at F. Points P and Q are the reflections of D over B and C.
   (a) Prove that points A, P, F, E, Q lie on a circle, say Γ.
   (b) The tangents to Γ at E and F meet at P. Prove that line AP passes through a fixed point as A varies.

Solution. For (a), we observe first that D is the orthocenter of $\triangle AEF$. Hence the *nine-point circle* of $\triangle AEF$ passes through B, C and the midpoints of AD, FD, ED. Thus Γ is the image of this nine-point circle under a homothety at D of ratio 2.

For (b), we claim that AP bisects $\overline{BC}$, which implies the result. Indeed, $\overline{AP}$ is an A-symmedian of $\triangle AEF$ and so it is isogonal to the A-median of $\triangle AEF$. On the other hand $\triangle ABC$ and $\triangle AEF$ are similar and oppositely oriented. So $\overline{AP}$ becomes the A-median of $\triangle ABC$, which is what we wanted to prove. □